PUPC Grading Scheme

November 17, 2017

General Guidelines

Error propagation: If an answer to a question is wrong but used consistently throughout the rest of the problem (and the calculations involving the wrong answer are correct) only deduct the points for the wrong answer and give full marks for the following answers even if they no longer match the answer key.

Dimensionally incorrect answers: If an answer isn't dimensionally correct give 0 points.

Incorrect constants: If an answer has the wrong numerical constants but the correct functional form, give half marks.

Problem 1: Warm-Up

a. For a general body with mass m, radius r, and moment of inertia I, the energy of the rolling body is:

$$E = I\omega^2/2 + mv^2/2 + mgh \qquad \text{(1 point)}$$

The bodies are rolling with out slipping, thus $v = \omega r$. Let α be the angle of the incline $\rightarrow h = ysin\alpha$, where y is the vertical position of the body with respect to the bottom of the incline. So:

$$E = \frac{1}{2}(I + mr^2\omega^2) + mgysin\alpha = \frac{1}{2}(I + mr^2)(v^2/r^2) + mgysin\alpha$$

Energy is conserved, so $\frac{dE}{dt} = 0$. Computing the derivative and recalling that $\frac{dy}{dt} = v$ and $\frac{dv}{dt} = a$ we get:

$$(\frac{I}{r^2} + m)va - mgv\sin\alpha = 0$$

$$a = \frac{mgsin\alpha}{I/r^2 + m} \rightarrow a = \frac{gsin\alpha}{I/mr^2 + 1} \qquad \text{(3 points)}$$

The first body to arrive at the bottom is the one with the greatest acceleration, which is equivalent to the body with the minimum I/mr^2.

For the hollow pipe: $\dfrac{I}{mr^2} = \dfrac{ma^2}{ma^2} = 1$

For the solid sphere: $\dfrac{I}{mr^2} = \dfrac{\frac{2}{5}m(2a)^2}{m(2a)^2} = \dfrac{2}{5}$

For the solid pipe: $\dfrac{I}{mr^2} = \dfrac{\frac{1}{2}(2m)a^2}{(2m)a^2} = \dfrac{1}{2}$

So the solid sphere will arrive first. (1 point)

Note: this problem can also be solved using torques.

b. Let the point have mass m. This problem is easily solved using the principle of superposition. Since the gravitational force is linear in mass, the point where the mass was removed can be thought of holding two masses - the initial mass m, plus a point mass of mass $-m$.

The field of the 2018 evenly space point masses is 0 at P. So the only component comes from the negative mass we introduced.

Suppose the negative mass is located at point Q; then $\overrightarrow{QP}$ be the vector connecting the negative mass to point P. Therefore:

$$\vec{\Gamma}_{(p)} = \frac{Gm}{R^2}\frac{\overrightarrow{QP}}{R} \qquad \text{(0.5 points)}$$

$$|\vec{\Gamma}_{(p)}| = \frac{Gm}{R^2} \qquad \text{(0.5 points)}$$

Four points are awarded for the explanation.

c. The points of equilibrium are given by $\dfrac{dU}{dx} = 0$. Therefore:

$$e^x(\sin x + \cos x) = 0 \to \sin x = -\cos x$$

$$x_1 = \frac{3\pi}{4}, x_2 = \frac{7\pi}{4} \qquad \text{(1 point)}$$

An equilibrium is stable if U is at a local minimum ($\dfrac{d^2U}{dx^2} > 0$) and unstable if U is at a local maximum ($\dfrac{d^2U}{dx^2} < 0$). Computing $\dfrac{d^2U}{dx^2} = 2e^x \cos x$ yields:

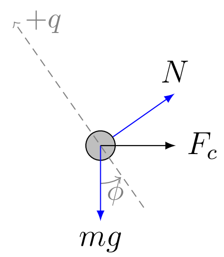
$$\cos(7\pi/4) = \sqrt{2}/2 \to 7\pi/4 \text{ is a stable equilibrium} \qquad \text{(1 point)}$$

Since $x \in [0, 2\pi)$, we also need to check the point 0 to see if it is a point of local minimum or maximum of U. (1 point for identifying $x = 0$ as a point of equilibrium.)

$$\frac{d^2U}{dx^2}(0) = 2e^0 \cos 0 = 2 > 0$$

Therefore, $x = 0$ is a point of local minimum, and the equilibrium is stable. (1 point)

Problem 2: Bead on Rotating Rod



a. The equation of motion parallel and perpendicular to the rod are respectively

$$m\ddot{q} = F_c \sin\phi - mg\cos\phi \tag{1}$$
$$0 = F_N - mg\sin\phi. \tag{2}$$

where F_N is the normal force and F_c is the centripetal force. Now

$$F_c = m\omega^2 R_0 = m\omega^2 q_0 \sin\phi. \tag{3}$$

By (1), since we require equilibrium at $\omega = \omega_c$,

$$m\ddot{q} = m(\omega^2 q_0 \sin\phi - g\cos\phi) = 0. \tag{4}$$

Thus

$$\omega_c^2 = \frac{g}{q_0 \tan\phi}. \tag{5}$$

Two points for using Newton's laws correctly. One point for correctly identifying the centripetal (or centrifugal) force. One point for stating equilibrium condition. One point for correct answer.

b. Fix $\omega > \omega_c$. Then from (1),

$$\ddot{q} - (\omega^2 \sin\phi)q + g\cos\phi = 0. \tag{6}$$

All (the whole three points) or nothing.

c. Plugging in the solution

$$q(t) = A_1 e^{\Omega t} + A_2 e^{-\Omega t} + B \tag{7}$$

we have

$$\Omega = \omega\sqrt{\sin\phi}, \quad B = \frac{g}{\omega^2 \tan\phi}. \tag{8}$$

As the bead starts from rest $\dot{q}(0) = 0$ implies

$$A_1 = A_2 = A. \tag{9}$$

As $q(0) = q_0 = A_1 + A_2 + g/(\omega^2 \tan\phi)$,

$$A = \frac{1}{2}\left[q_0 - \frac{g}{\omega^2 \tan\phi}\right]. \tag{10}$$

Compactly

$$q(t) = 2A\cosh(\Omega t) + B.$$

One point awarded for each correct determination of unknown constant. Two points for determining Ω. The use of hyperbolic functions is not necessary, and the test-taker will not be penalized for not using them.

d. The bead flies off when

$$q(t_f) = L = 2A\cosh(\Omega t_f) + B = A(e^{\Omega t_f} + e^{-\Omega t_f}) + B \tag{11}$$

or

$$t_f = \cosh^{-1}\Big(\frac{L-B}{2A}\Big). \tag{12}$$

One point for writing down either of the above equations (or equivalent).

e. When the bead flies off, it will be launched at an angle ϕ with respect to the vertical with a component of velocity parallel to the rod

$$v = \dot{q}(t_f) = 2A\Omega\sinh(\Omega t_f) = A\Omega(e^{\Omega t_f} - e^{-\Omega t_f}). \tag{13}$$

But because the rod is spinning, the bead will also be launched with a tangential component of velocity

$$\omega R = \omega L\sin\phi. \tag{14}$$

Two points for each component of velocity above correctly identified.

f. Let $v = \dot{q}(t_f)$ (see (13)). Then we want to solve for when the bead hits the ground:

$$L\cos\phi + (v\cos\phi)t - \frac{gt^2}{2} = 0. \tag{15}$$

The quadratic formula gives

$$T = \frac{1}{g}(-v\cos\phi + \sqrt{v^2\cos\phi + 2gL\cos\phi}). \tag{16}$$

Two points for each equation.

g. In one component, the bead will travel a distance of $(v\sin\phi)T$ away from the launch point. The launch point itself is $L\sin\phi$ away from the center, so in one component, the bead will travel

$$(v\sin\phi)T + L\sin\phi.$$

However, because the rod is spinning, the bead will also have a tangential component of velocity ωR (see part e)). Thus, the bead will be deflected a perpendicular distance of

$$\omega RT.$$

In total, (by the Pythagorean theorem) the bead will land at a distance of

$$\sqrt{(\omega RT)^2 + [(v\sin\phi)T + L\sin\phi]^2} \tag{17}$$

from the center. *All (the full three points) or nothing.*

Problem and solution written by Alex Chen.

Problem 3: Disk Oscillations

Part 1

a.

$$\int \rho(r)dA = \int_0^R \rho(r)2\pi r dr = \int_0^R kr(2\pi r)dr$$
$$= 2\pi k \int 0^R r^2 dr = \frac{2\pi}{3}kR^3 \qquad \text{(3 points)}$$

b. By definition,

$$I = \int r^2 dm = \int_0^R \rho(r)(2\pi r)r^2 dr = 2\pi k \int_0^R r^4 dr$$
$$= \frac{2\pi}{5}kR^5 = \frac{2\pi}{3}kR^3 \times \frac{3}{5}R^2 = \frac{3}{5}MR^2 \qquad \text{(5 points)}$$

c. By symmetry, the center of mass will be the geometrical center of the disk. The energy of oscillations is:

$$E = I\dot{\theta}^2/2 + mv^2/2 + Mgh$$

Plugging in $v = R\dot{\theta}$ and $h = R(1-\cos\theta)$ yields:

$$E = \frac{1}{2}(I + MR^2)\dot{\theta}^2 + MgR(1-\cos\theta) \qquad \text{(1 point)}$$

Note: the first term can also be found by applying the Parallel Axis Theorem.

$$\text{Energy is conserved} \rightarrow \frac{dE}{dt} = 0 \qquad \text{(1 point)}$$

Taking the derivative and simplifying yields:

$$(I + MR^2)\dot{\theta}\ddot{\theta} + MgR\sin\theta\dot{\theta} = 0$$
$$\ddot{\theta} = -\frac{MgR}{I + MR^2}\sin\theta \qquad \text{(1 point)}$$

Since $\sin\theta \approx \theta$ for small angles, we can rewrite this second-order differential equation in a recognizable form:

$$\ddot{\theta} = -\frac{MgR}{I + MR^2}\theta = -\omega^2\theta$$

Therefore the frequency of oscillations is:

$$\omega = \sqrt{\frac{MgR}{I + MR^2}} = \sqrt{\frac{MgR}{\frac{8}{5}MR^2}} = \sqrt{\frac{5g}{8R}} \qquad \text{(2 points)}$$

Note: this problem can also be solved using torques.

Part 2

a. Upper half:

$$\frac{M}{2} = \int_0^R \rho_1 \pi r dr = \rho_1 \frac{\pi R^2}{2} \rightarrow \rho_1 = \frac{M}{\pi R^2}$$

Lower half:

$$\frac{M}{2} = \int_0^R \rho_2 \pi r dr = k \int_0^R \pi r^2 dr = \pi k \frac{R^3}{3} \rightarrow k = \frac{3M}{2\pi R^3}$$

Now we compute the moments of inertia of the respective halves with respect to the center C:

$$\begin{aligned} I_{\text{upper}} &= \int_0^R \rho_1 \pi r (r^2) dr = \rho_1 \pi \int_0^R r^3 dr \\ &= \rho_1 \pi \frac{R^4}{4} = \frac{M}{\pi R^2} \times \frac{\pi R^4}{4} = \frac{MR^2}{4} \end{aligned} \qquad \text{(1 point)}$$

$$\begin{aligned} I_{\text{lower}} &= \int_0^R \rho_2 \pi r (r^2) dr = k\pi \int_0^R r^4 dr \\ &= k\pi \frac{R^5}{5} = \frac{3M}{2\pi R^3} \times \frac{\pi R^5}{5} = \frac{3}{10} MR^2 \end{aligned} \qquad \text{(2 points)}$$

To account for the circular hole, we can superimpose a disk of "negative mass" $-M/4$. Invoking the Parallel Axis Theorem, this yields:

$$I_{\text{hole}} = -\frac{1}{2}\frac{M}{4}\left(\frac{R}{2}\right)^2 - \frac{M}{4}\left(\frac{R}{2}\right)^2 = -\frac{3}{32} MR^2$$

Summing these moments of inertia, we find:

$$I = I_{\text{upper}} + I_{\text{lower}} + I_{\text{hole}} = \frac{73}{160} MR^2 \qquad \text{(3 points)}$$

b. Please see the attached PDF for a step-by-step solution to this problem.

Problem 4: Soap Bubble

Equilibrium of forces at detachment: $F_{\text{weight}} + F_{\text{air flow}} = F_{\text{surface tension}}$

$$F_{\text{weight}} = (\text{mass of soap})g = \sigma g[4\pi R^2 - \pi(r^2 + h^2)] \qquad \text{(2 points)}$$

$$\begin{aligned} F_{\text{air flow}} &= (\text{number of molecules going into the bubble per unit time}) \\ &\qquad \times \text{ change of momentum/unit time} \\ &= \rho\pi r^2 v\Delta t \frac{\Delta v}{\Delta t} = \rho\pi r^2 v\Delta v = \pi\rho r^2 v^2 \end{aligned} \qquad \text{(7 points)}$$

$$\begin{aligned} F_{\text{surface tension}} &= \frac{\Delta E}{\Delta z} = \frac{\Delta((\gamma A)}{\Delta z} \\ &= \frac{(2)(2\pi r)\gamma\Delta z}{\Delta z} = 4\pi r\gamma \end{aligned} \qquad \text{(8 points)}$$

The factor of 2 comes from the fact that there are 2 soap films around the bubble. The equation becomes:

$$\begin{aligned} 4\pi\sigma g R^2 - \pi\sigma g r^2 - \pi\sigma g h^2 + \pi\rho r^2 v^2 &= 4\pi r\gamma \\ 4\sigma g R^2 - \sigma g(R - \sqrt{R^2 + r^2})^2 - \sigma g r^2 + \rho r^2 v^2 &= 4r\gamma \\ \sigma g(4R^2 - R^2 - R^2 + r^2 + 2R\sqrt{R^2 - r^2}) &= 4r\gamma + \sigma g r^2 - \rho r^2 v^2 \end{aligned} \qquad \text{(2 points)}$$

The volume of the bubble after detachment equals the volume of the "sphere with cap removed" with radius R found above. That is:

$$\frac{4}{3}\pi R_f^3 = \frac{4}{3}\pi R^3 - \frac{1}{3}\pi h^2(3R - h) \qquad \text{(2 points)}$$

Note: the above two equations do not have to be explicitly solved for full credit.