2014 WMI Mini Lecture Outlines

2014 WMI Mini Lecture
Mathematical Biology
Introduction to Population Genetics

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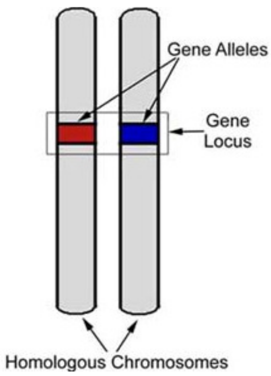
1. Basic Principles and Vocabulary in Genetics

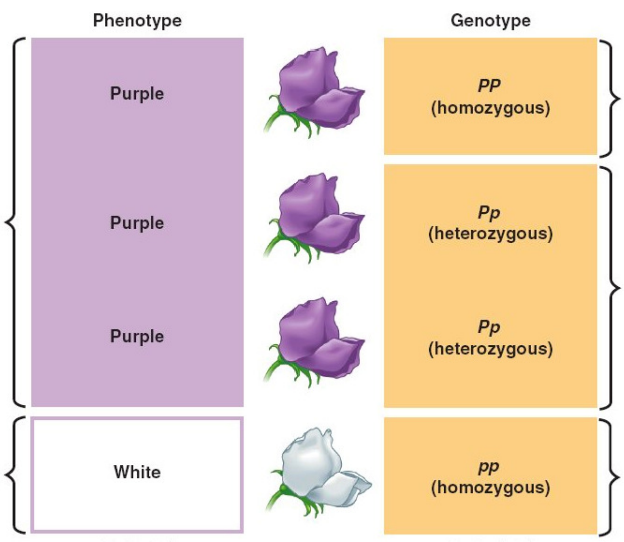
- Genes and Alleles

Genes are molecular units of heredity of a living organism.
Allele is one of alternative forms of the same gene or same genetic locus.



- Genotype and Phenotype

Genotype: outward, physical manifestation; composes of two alleles
Phenotype: internally coded, inheritable information

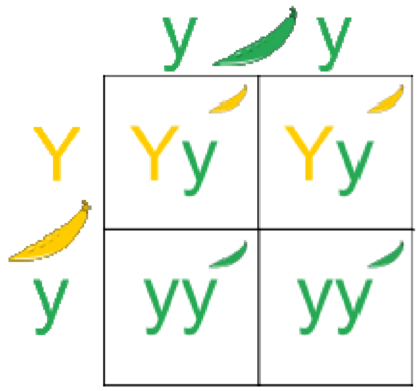


- Dominant and Recessive Alleles

  Dominance in alleles determines the expression of traits. Each allele could be either dominant or recessive. If a dominant allele exists in a gene, the phenotype follows the dominant form. If there isn't any dominant allele but only recessive alleles, the phenotype shows the recessive form.

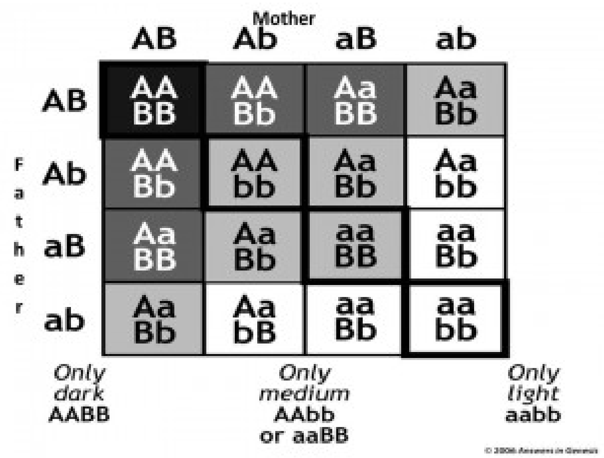
- Punnet Square

  PunnetSquare is a simple grid with all possible male gamete genes and female gamete genes. The grid can be completed by filling in square with the genotype that can be generated from each combination of the row and the column.



2. Basic Math in Genetics

- Possibilities of Heredity



- Hardy – Weinberg Principle

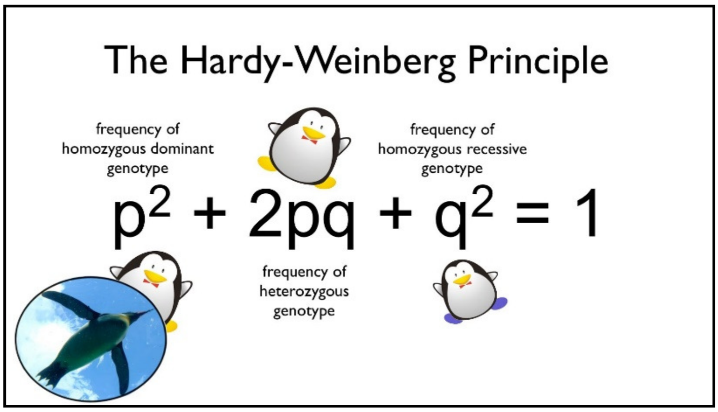
1) Requirements

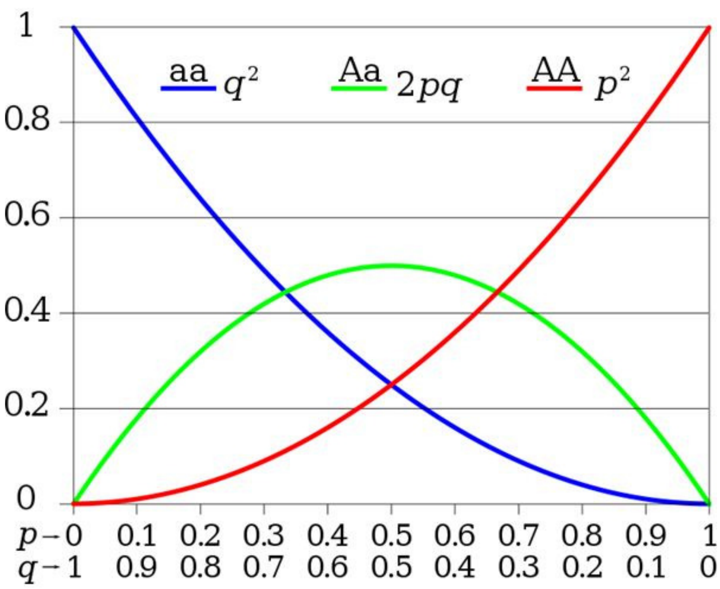
   a) No selection
   b) No mutation
   c) Infinitely large population
   d) No emigration or immigration
   e) Random mating

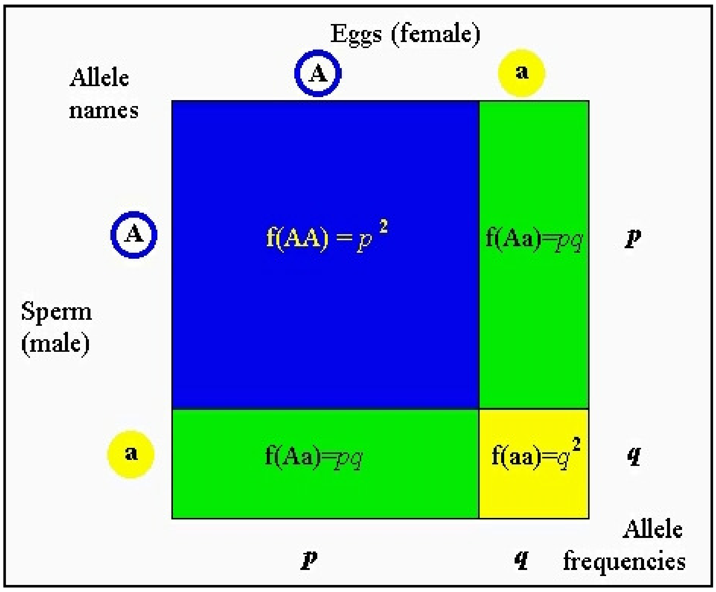
2) Hardy – Weinberg Equilibrium

   When
   p= frequency of ‘A allele’
   q= frequency of ‘a allele’

$$p + q = 1$$
$$(p + q)^2 = 1$$
$$p^2 + 2pq + q^2 = 1$$







3. Effects of Selection

I. Effects of Selection

p_t= Relative frequency of ***A*** allele in current generation ($0 \leq p_t \leq 1$)

q_t= Relatice frequency of ***a*** allele in current generation ($0 \leq q_t \leq 1$)

w_{AA}= Relative survival of AA genotype

w_{Aa}= Relative survival of Aa genotype

w_{aa}= Relative survival of aa genotype

N= The real population size

Genotype frequency= Different from a frequency of an allele, it means the frequency of genotype itself, such as Aa or AA.

According to Hardy – Weinberg Equilibrium, in current generation *t*,

$$q_t + p_t = 1$$

$$q_t^2 + 2p_tq_t + p_t^2 = 1$$

After the time goes on, in the next generation *t+1,* the relative frequencies will change, so let’s multiply each genotype frequency by its relative survival.

$$\text{AA}: p_t^2 w_{AA}$$
$$\text{Aa}: 2p_t q_t w_{Aa}$$
$$aa: p_t^2 w_{aa}$$

(1)

The relative frequencies will have relationships below.

$$p_{t+1} + q_{t+1} = 1$$

Average survival $\overline{w_t}$ in *t+1* will be

$$\overline{w_t} = p_t^2 w_{AA} + 2p_t q_t + q_t^2 w_{aa}$$

∴The new relative genotype frequencies, whose sum is out of 1, will be the frequencies in (1) divided by the average survival $\overline{w_t}$.

$$AA: p_t^2 \frac{w_{AA}}{\overline{w_t}}$$
$$Aa: 2p_t q_t \frac{w_{Aa}}{\overline{w_t}}$$
$$aa: q_t^2 \frac{w_{aa}}{\overline{w_t}}$$

(2)

Meanwhile, the frequencies in the next generation *t+1* can be determined, using similar principle in Punnet Square.

$$p_{t+1} = \frac{2N \cdot frequency\ of\ AA + N \cdot frequncy\ of\ Aa}{2N}$$
$$= frequency\ of\ AA + \frac{1}{2}(frequency\ of\ \ Aa)$$
$$= \frac{p_t^2 w_{AA} + \frac{1}{2} \cdot 2p_t q_t w_{Aa}}{\overline{w_t}} = \frac{p_t(p_t w_{AA} + q_t w_{Aa})}{\overline{w_t}}$$

Define Δ_p as the change in *p* from generation to *t+1,*

$$\Delta_p = p_{t+1} - p_t = \frac{p_t(p_t w_{AA} + q_t w_{Aa})}{\overline{w_t}} - p_t$$
$$= \frac{p_t}{\overline{w}}(p_t w_{AA} + q_t w_{Aa} - \overline{w}_t)$$
$$= \frac{p_t}{\overline{w}}[p_t w_{AA} + q_t w_{Aa} - (p_t^2 w_{AA} + 2p_t q_t + q_t^2 w_{aa})]$$
$$= \frac{p_t}{\overline{w_t}}[w_{AA} \cdot p_t(1-p_t) + w_{Aa} \cdot q_t(1-2p_t) - {q_t}^2 \cdot w_{aa}$$
$$= \frac{p_t}{\overline{w_t}}[w_{AA} \cdot p_t q_t + w_{Aa} q_t(1-p_t-p_t) - {q_t}^2 \cdot w_{aa}]$$
$$= \frac{p_t}{\overline{w_t}}[w_{AA} \cdot p_t q_t + q_t(q_t-p_t) \cdot w_{Aa} - q\_t^{\ 2} \cdot w_{Aa}]$$
$$= \frac{p_t q_t}{\overline{w_t}}[w_{AA} \cdot p_t + (q_t - p_t) \cdot w_{Aa} - q_t \cdot w_{Aa}]$$
$$= \frac{p_t q_t}{\overline{w_t}}[p_t \cdot (w_{AA} - w_{Aa}) + q_t \cdot (w_{Aa} - w_{aa})]$$

$$\Delta_p = \frac{p_t q_t}{\overline{w}}[p_t \cdot (w_{AA} - w_{Aa}) + q_t \cdot (w_{Aa} - w_{aa})] \qquad (3)$$

From equation (2) above,

$$\overline{w} = {p_t}^2 w_{AA} + 2p_t q_t w_{Aa} + q_t^2 w_{aa}$$

To express *q* in terms of *p,*

$$\overline{w} = p_t^2 w_{AA} + 2p_t(1-p_t)w_{Aa} + (1-p_t)^2 w_{aa}$$

$$= p_t^2 w_{AA} + 2p_t w_{Aa} - 2p_t^2\, w_{Aa} + w_{aa} - 2p_t w_{aa} + p_t^2 w_{aa}$$

Then take the derivative of this to get an expression for how the average fitness of the population changes as the allele frequencies change.

$$\frac{d}{dp}\overline{w} = 2p\, w_{AA} + 2w_{Aa} - 4p w_{Aa} + 0 - 2w_{aa} + 2p\, w_{aa}$$

$$= 2(p\, w_{AA} + w_{Aa} - 2p\, w_{Aa} - w_{aa} + p w_{aa})$$

$$= 2[p(w_{AA} - W_{Aa}) + (1-p)(w_{Aa} - w_{aa})]$$

(4)

However, note that $\Delta_p = \frac{pq}{\overline{w}_t}[p_t(w_{AA} - w_{Aa}) + q_t(W_{Aa} - w_{aa})]$

Then,

$$\Delta_p = \frac{p_t q_t}{2\overline{w_t}} \cdot \frac{d}{dp}\overline{w_t}$$

(5)

According to (5),

While pq is proportional to the variance in allele frequencies,

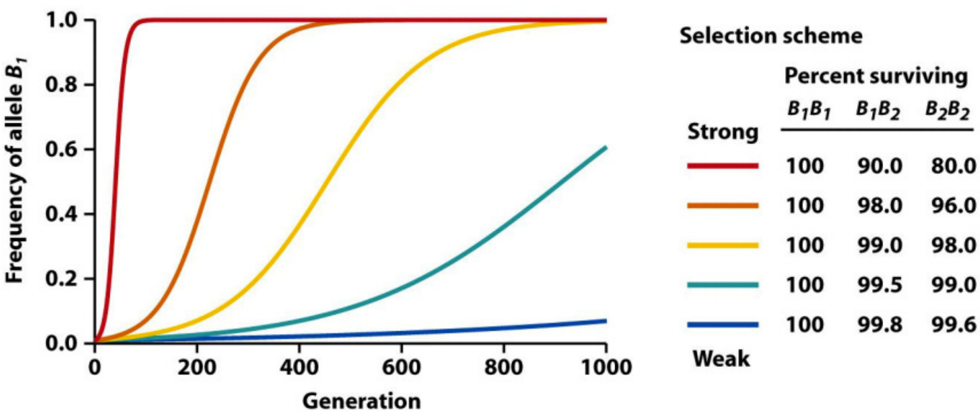
$\frac{d}{dp}\overline{w_t}$ shows the strength of selection: how strongly fitness varies.

The rate of evolution is proportional to the product of

Genetic variability ×strength of selection

At the non-trivial equilibrium, with both p and q greater than 0, $\Delta_p = 0$ requires that $\frac{d}{dp}\overline{w} = 0$.

∴Natural selection maximizes the mean fitness$\overline{w}$.



2014 WMI Mini Lecture
24 Games

Hong Kong Mathematical Olympiad School
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August 10, 2014

遊戲名稱：合24
負責單位：香港數學奧林匹克學校
遊戲玩法：

運用所提供的四個數字及運算符號+、-、×、÷、( )及$\sqrt{\ }$，得出一道答案等於24的算式。
(每個數字都必須用，次序可調換。)

例子：	244 9
算式：	$(4-2)\times\sqrt{9}\times 4$

快來參與，成功解答者，可得小禮物一份！

24 Game
By: Hong Kong Mathematical Olympiad School

How to play:

Make 24 by using four given digits and operations such as +、-、×、÷、( ) and $\sqrt{\ }$.
(All given digits must be used and their orders can be changed.)

For example：	244 9
Combinations：	$(4-2)\times\sqrt{9}\times 4$

Come and join us. Student who solves the problem successfully will receive a gift.