# Introduction to Functions

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## 1. What is a function?

A function is a "relation" between the sets that associate one input to a single output. The set of input values of the function is called the "domain" and the set of output values of the function is called the "range." Therefore, one value in the domain should correspond to a single value in the range as depicted in **Figure 1**.

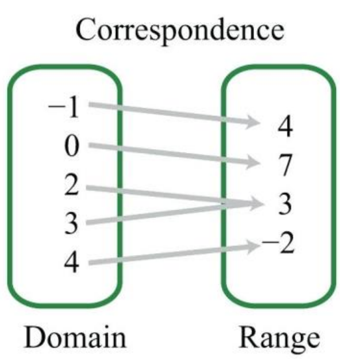


**Figure 1**: Correspondence between the domain and the range of the function.

## 2. One-to-one function

One-to-one function is a function where each input has a single unique output. Therefore, the function depicted in **Figure 1** is not a one-to-one function since 2 and 3 in the domain have the same output in the range. **Figure 2** depicts one-to-one function.

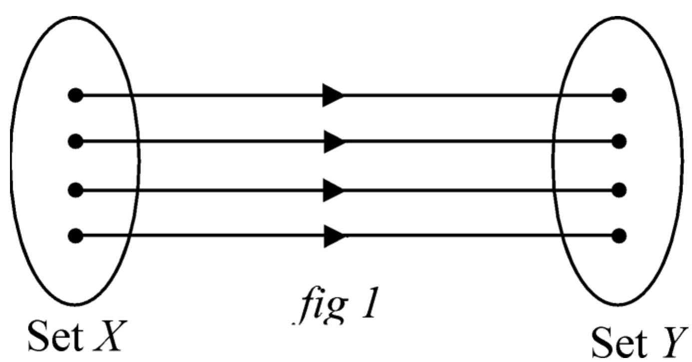


**Figure 2**: Correspondence between the domain and the range of one-to-one function.

## 3. Vertical line test and horizontal line test

In order to determine whether a relation is a function or even one-to-one function, there are two types of tests that can be used. First of all, vertical line test is used in order to determine whether a relation is a function. The following is the graph of the function, $y = x$.

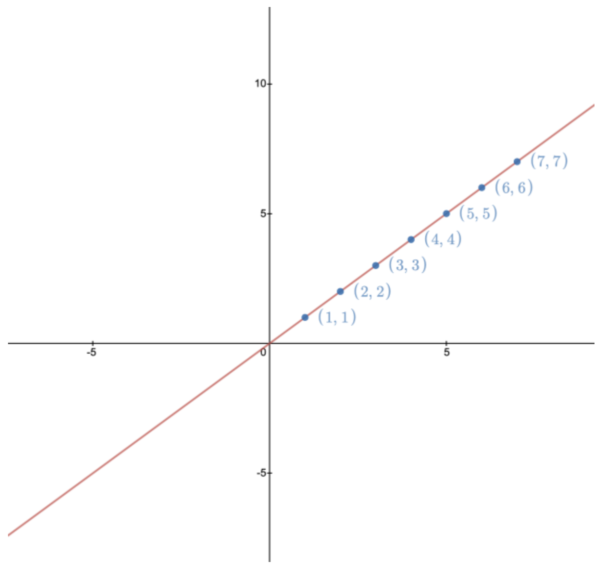


**Figure 3**: The graph of $y = x$.

If it is assumed that there are multiple imaginary vertical lines that are parallel to the $y$ axis, it is clear that each vertical line intersects the function, $y = x$ at only one point. This also means that for each value in the domain, there is only one output associated to it in the range which is the exact definition of a function. Thus, it is safe to say that if the graph passes the vertical line test, it is a function.

However, passing the vertical line test is not sufficient to prove whether the function is one-to-one. Horizontal line test is needed to prove this. Again, think of the function, $y = x$ and assume there are multiple imaginary horizontal lines that are parallel to the $x$ axis. In this case also, each horizontal line intersects the function at only one point. This signifies that for each value in the domain, there is only one unique output associated to it in the range which is the definition of a one-to-one function. Therefore, if the function passes both the vertical line test and the horizontal line test, it is a one-to-one function.

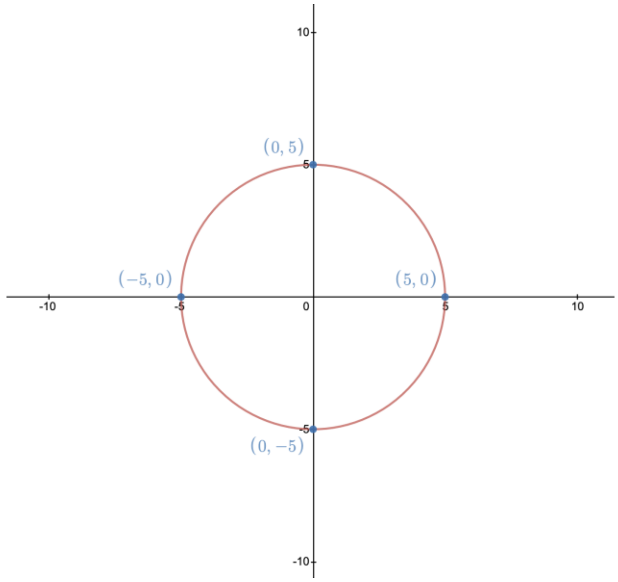


**Figure 4**: The graph of $x^2 + y^2 = 25$.

The function like $x^2 + y^2 = 25$ as shown in **Figure 4** is not one-to-one nor a function because it fails to pass both the vertical line test and the horizontal line test.

### 4. Inverse functions

Inverse function is a "reverse" of the original function. If the function $f$ has an input $x$ and output $y$, the inverse function $f^{-1}$ has an input $y$ and output $x$. This also means that the domain and the range are switched between the original function and the inverse function.

Knowing this, it is also possible to derive the equation of the inverse function. Simply switching $x$ and $y$ in the equation works because the input and output is switching. For instance, it will work like this.

- $y = 2x + 5$
- $x = 2y + 5$
- $x - 5 = 2y$
- $y = \frac{1}{2}x - \frac{5}{2}$
- $f^{-1}(x) = \frac{1}{2}x - \frac{5}{2}$

It is also possible to find out the inverse function by using the graph as shown in **Figure 5**.

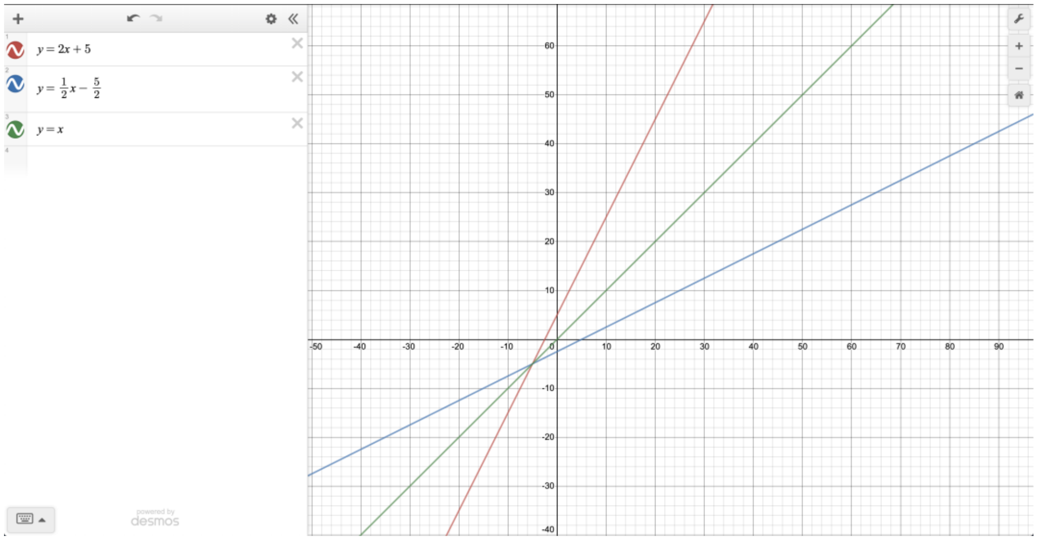


**Figure 5**: Relationship between the original function and its inverse function.

From **Figure 5**, it is confirmed that the graph of the original function and its inverse function are mirror images of each other over the graph of $y = x$. This makes sense because finding an inverse function requires the switch between the input and the output and this is effectively the same as reflecting the original function over $y = x$.

There are multiple ways to determine whether a function has its inverse function. For its inverse function to be a function in the first place, the graph of the inverse function should pass the vertical line test. Knowing that the inverse function is a mirror image of the original function over $y = x$, this means that the original function should pass the horizontal line test.

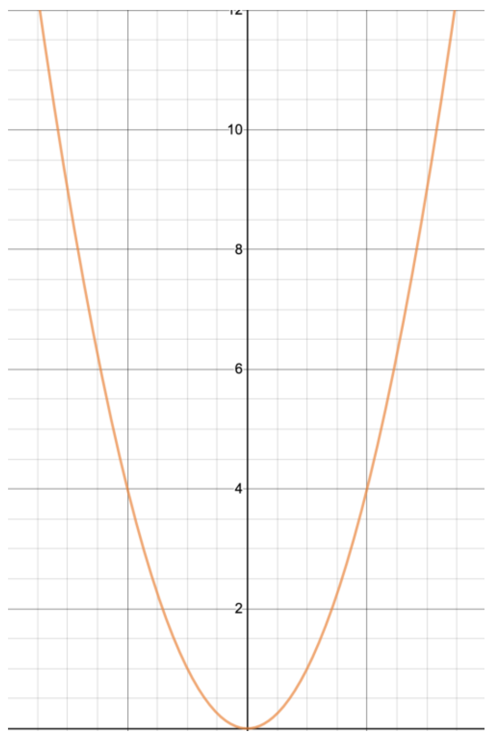


**Figure 6**: The graph of $y = x^2$.

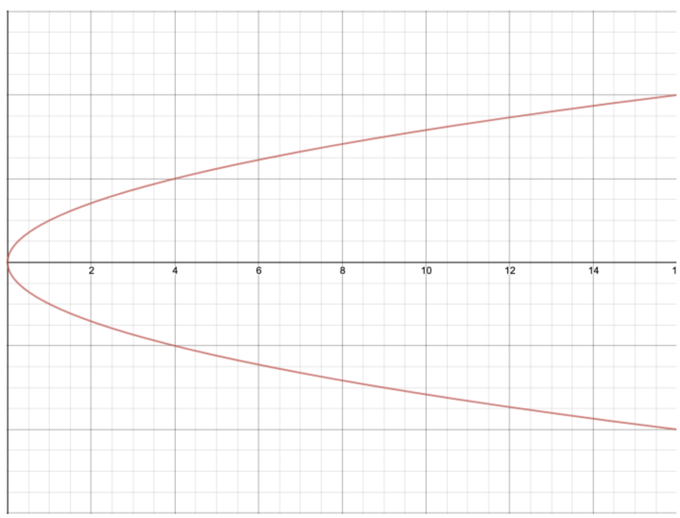


**Figure 7**: The graph of $y = \pm\sqrt{x}$.

For instance, $y = \pm\sqrt{x}$ is an inverse of $y = x^2$. As shown in **Figure 7**, $y = \pm\sqrt{x}$ is not a function since it doesn't pass the vertical line test. This means that $y = x^2$ doesn't pass the horizontal line test which is clearly shown in **Figure 6**. This indicates that if the original function is not one-to-one, its inverse is not a function.

## 5. Transformations of functions

There are three types of transformations: translation, reflection, and dilation.

- Translation
    - The curve is shifted
- Reflection
    - The curve is flipped
- Dilation
    - The curve gets smaller/bigger

To easily perform transformations of quadratic functions, it is important to change the standard form into the vertex form.

- Standard form
    - $y = ax^2 + bx + c$
- Vertex form
    - $y = a(x-h)^2 + k$
        - $(h, k) \Rightarrow vertex$

For each type of transformation, it is easier to understand with examples. To start, the following is an example of horizontal translation.

*Horizontal translation to the right by 4 units*

- $y = (x-2)^2 + 4$
    - The vertex has to change from $(2,4)$ to $(6,4)$
    - $y = (x-6)^2 + 4$

*Horizontal translation to the left by 4 units*

- $y = (x-2)^2 + 4$
    - The vertex has to change from $(2,4)$ to $(-2,4)$

- $y = (x + 2)^2 + 4$

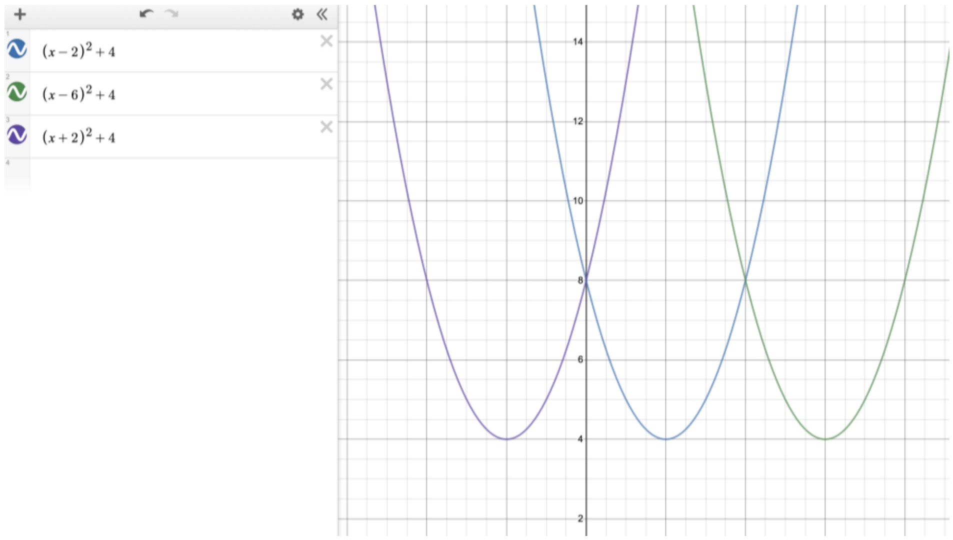


**Figure 8**: Horizontal translation of the function.

The following is an example of vertical translation.

*Vertical translation up by 4 units*

- $y = (x - 2)^2 + 4$
  - The vertex has to change from (2,4) to (2,8)
  - $y = (x - 2)^2 + 8$

*Vertical translation down by 4 units*

- $y = (x - 2)^2 + 4$
  - The vertex has to change from (2,4) to (2,0)
  - $y = (x - 2)^2$

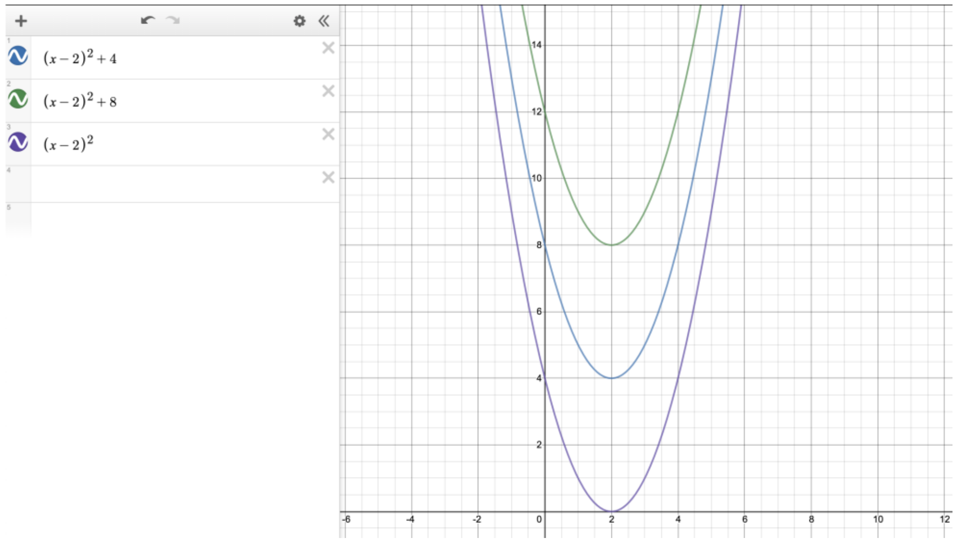


**Figure 9**: Vertical translation of the function.

The following is an example of reflections.

*Reflection over x-axis*

- $y = (x-2)^2 + 4$
  - The $y$ values are changing signs
  - $y = -(x-2)^2 - 4$

*Reflection over y-axis*

- $y = (x-2)^2 + 4$
  - The $x$ values are changing signs
  - $y = (-x-2)^2 + 4$

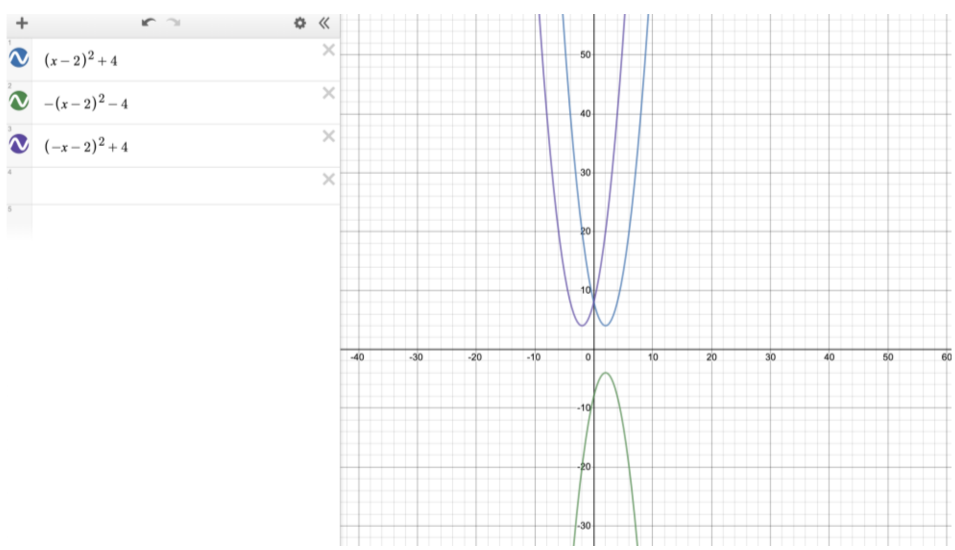

**Figure 10**: Reflections of the function.

The following is an example of horizontal dilation.

*Horizontal dilation by a scale factor 2*

- $y = x^2$
  - The graph is stretching horizontally by 2
  - $y = \left(\frac{1}{2}x\right)^2 = \frac{1}{4}x^2$

*Horizontal dilation by a scale factor ½*

- $y = x^2$
  - The graph is shrinking horizontally by ½
  - $y = (2x)^2 = 4x^2$

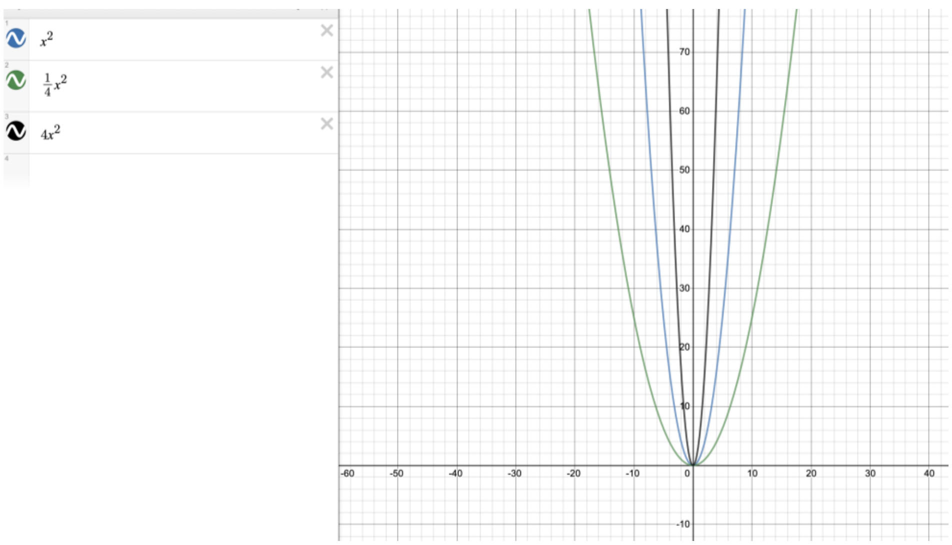


**Figure 11**: Horizontal dilation of the function.

Lastly, the following is an example of vertical dilation.

*Vertical dilation by a scale factor 2*

- $y = x^2$
  - The graph is stretching vertically by 2
  - $y = 2x^2$

*Vertical dilation by a scale factor ½*

- $y = x^2$
  - The graph is shrinking vertically by ½
  - $y = \frac{1}{2}x^2$

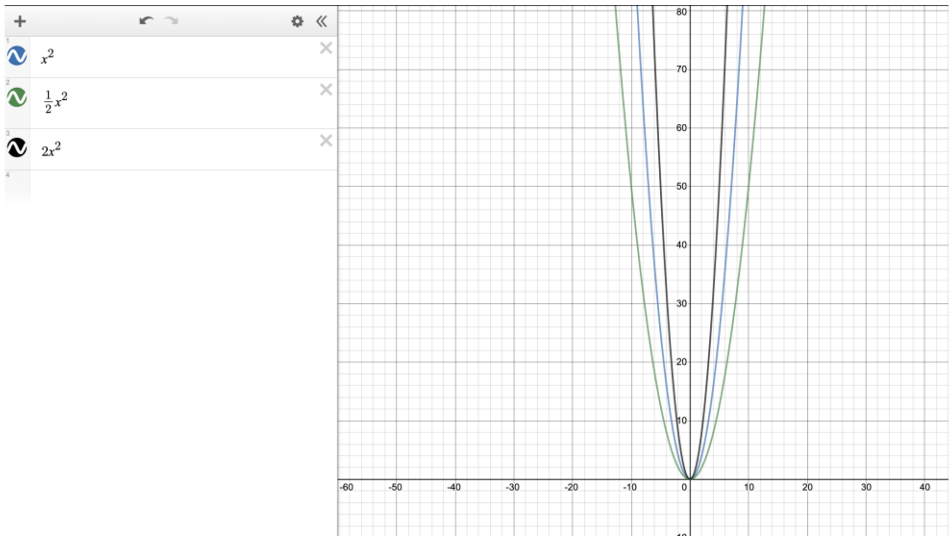


**Figure 12**: Vertical dilation of the function.