Vieta's Formulas

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1. Introduction

Vieta's Formulas are a set of formulas developed by the French Mathematician Franciscus Vieta that relates the sum and products of roots to the coefficients of a polynomial.

We begin by understanding how Vieta's formulas may be useful.

Find the roots of the following equations and find the sum and product of their roots:

A. $x^2 - 1 = 0$
B. $x^2 + 3x - 18 = 0$
C. $5x^2 + 10x - 40 = 0$
D. $6x^3 - 36x^2 + 66x - 36 = 0$

Solutions:

A. $x^2 - 1 = 0, (x + 1)(x - 1) = 0, x = 1 or - 1$
   a. $Product\ of\ roots = -1$
   b. $Sum\ of\ the\ roots = 0$
B. $x^2 + 3x - 18 = 0, (x + 6)(x - 3) = 0, x = 3\ or\ - 6$
   a. $Product\ of\ roots = -18$
   b. $Sum\ of\ the\ roots = -3$
C. $5x^2 + 10x - 40 = 0,5(x + 4)(x - 2) = 0, x = 2\ or\ - 4$
   a. $Product\ of\ roots = -8$
   b. $Sum\ of\ the\ roots = -2$
D. $6x^3 - 36x^2 + 66x - 36 = 0,6(x - 1)(x - 2)(x - 3) = 0, x = 1,2,3$
   a. $Product\ of\ roots = 6$
   b. $Sum\ of\ the\ roots = 6$

It is possible to solve each equation for the value of their roots and then finding the sum and product of the roots. However, solving for the exact value of the roots can be difficult for some equations. Consider the following equations.

A. $x^{10} - 55x^9 + 1320x^8 - 18150x^7 + 157773x^6 - 902055x^5 + 3416930x^4$

$$-8409500x^3 + 12753576x^2 - 10628640x + 362880 = 0$$

B. $x^3 - 9x^2 + ax - 24 = 0$

For the following equation, it is hard to find the value of the roots. However, what if there was a method to find the value of the sum and product of the roots directly?

2. Derivation of Vieta's formula in a quadratic equation

To answer this question, we start off with finding the sum and product of the roots of a generalised quadratic equation.

Given quadratic $ax^2 + bx + c = 0$, find the sum and products of the roots of the equation

By the fundamental theorem of algebra, this can be written in the form:

$$a(x - r)(x - s) = 0$$

where r and s stand for the two roots of the equation. Expanding out the equation above gives us:

$$ax^2 - a(r + s)x + a(rs)x = 0$$

We can see that the value of b in the original equation corresponds to $-a(r + s)$and that c corresponds to $a(rs)$.

In equation form $b = -a(r + s)$and $c = a(rs)$,which means that $r + s = -b/a$ and $rs = c/a$.

Therefore,

$$Sum\ of\ the\ roots\ = -b/a$$

$$Product\ of\ the\ roots\ = c/a$$

3. Alternate proof of Vieta's Formula using the quadratic formula

The answer above can also be proved through the quadratic formula.

Given $ax^2 + bx + c = 0$, the roots of the equation are

$$\frac{-b+\sqrt{b^2-4ac}}{2a} \text{and} \frac{-b-\sqrt{b^2-4ac}}{2a}$$

Adding these two roots gives us,

$$\frac{-2b + \sqrt{b \quad - 4ac} - \sqrt{b \quad - 4ac}}{2a} = \frac{-b}{a}$$

and multiplying the roots gives us

$$\frac{-b^2 + b^2 - 4ac}{4a^2} = \frac{c}{a}$$

4. Derivation of Vieta's formulas in a polynomial equation

Vieta's formulas can also be applied to a polynomial of larger degrees.

Given a polynomial of degree n:

$$a_0x^n + a_1x^{n-1}...+a_nx^0$$

according to the Fundamental Theorem of Algebra the polynomial can be written as:

$$a(x - r_1)(x - r_2)(x - r_3)...(x - r_n \quad)$$

Expanding out this equation gives us:

$$ax^n - a(r_1 + r_2...+r_{n)}x^{n-1}+...+a(-1)^n(r_1r_2...r_n)$$

We can see that the value of $a_1$corresponds to $-a(r_1 + r_2...+r_n)$and that $a_n$corresponds to $a(-1)^n(r_1r_2...r_n)$. Setting up an equation and manipulating it gives us $r_1 \quad + r_2...+r_3 = \frac{-a_1}{a}$and $r_1r_2...r_n = (-1)^n\frac{a_n}{a}$. Therefore

$$Sum\ of\ the\ roots\ = -a_1/a$$

$$Product\ of\ the\ roots\ = (-1)^na_n/a$$

5. Combinatorial Proof for Veita's formulas

In the section above, we have proved Vieta's formulas using algebra. But the formula can also be conceptualised using a combinatorial approach. Consider the following polynomial with roots r_1 through r_n.

$$a(x-r_1)(x-r_2)(x-r_3)\ldots(x-r_n)$$

Consider each binomial enclosed by parenthesis as an “element” in this equation. When expanding out this equation, you can only multiply one part of element (either the x or the root) at a time to get a term due to the distributive property.

For example, in the following polynomial with two elements

$$(x-a)(x-b)$$

You must select one part of each element to multiply out and get a term. Therefore since there are two elements in the equation, there are four possible terms you can get from the equation: $x^2, -ax, -bx, ab$, and the coefficient of term with a specific degree in the fully expanded equation is the sum of all possible terms you can get using this method.

For example, the coefficient on the term with degree 1 is $(-a)+(-b)$, which is $-(a+b)$. This is consistent with the result you get when expanding the equation out manually.

Applying this observation back to the original observation, in order to get a term with degree $n-1$, you must “choose” the x part of an element for $n-1$ elements and you must choose the root part of the element for 1 element. This means that there are a total of $nC1$ terms with the degree $n-1$. Adding all these terms up gives us the coefficient of the term with degree $n-1$ in the fully expanded equation, and since you can only choose one root term, we know that the coefficient of the term with degree $n-1$ is the negative sum of all roots of the equation.

Similar insight can be used to figure out the product of all roots of an equation. In order to get a constant, or a term with degree 0, you must select the root part of all elements. There are $nCn = 1$ ways to do this, and therefore the coefficient of the constant term in an equation is always the $the\ product\ of\ all\ roots \times (-1)^n$.

6. Application of Vieta’s Formulas

With our new knowledge of Vieta’s formulas, we can solve now solve the problems that we couldn’t solve earlier.

A. $x^{10}-55x^9+1320x^8-18150x^7+157773x^6-902055x^5+3416930x^4$

$$-8409500x^3+12753576x^2-10628640x+362880=0$$

$$Sum\ of\ roots = (-1)\times -55/1 = 55$$

$$Product\ of\ roots\ =\ (-1)^{10} \times 3628800/1 = 3628800$$

B. $x^3 - 9x^2 + ax - 24 = 0$

$$Sum\ of\ roots\ =\ (-1) \times -9/1\ = 9$$

$$Product\ of\ roots\ =\ (-1)^3 \times -24/1\ = 24$$

7. Conclusion

Through this investigation, we have explored what vieta's formulas are, a number of proofs for the formulas, and applied the formulas to solve questions. Vieta's formulas are useful whenever you are finding the sum or products of roots of an equation. Vieta's formulas are also useful when finding the coefficient of a term with a specific degree, and similar methods of proof can be used to derive formulas for terms with degrees other than n-1 or 0.