Onsite Test 2015: Limited Solutions

December 6, 2015

1 Problem 1

a. The initial energy is

$$E_i = \frac{1}{2}kx^2$$

and the final energy is

$$E_f = mg(2R - 2r) + \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2$$

For the ball not to lose contact,

$$\frac{mv^2}{R} = mg.$$

Using conservation of energy, it follows that

$$x_{\min} = \sqrt{\frac{54mg}{10k}(R - r)}$$

b. A simple application of conservation of energy yields

$$\frac{1}{2}kx^2 = mgH,$$

or

$$H = \frac{27}{10}(R - r).$$

2 Problem 2

Note that the moment of inertia of a uniformly distributed triangle with mass M and distance from center to vertex L is $I = \frac{1}{4}ML^2$.

a. We apply conservation of angular momentum (about the center of the triangle), linear momentum, and energy. This gives

$mv = mv' + MV$ (1)

$mvL = Iw + mv'L$ (2)

$mv^2 = mv'^2 + Iw^2 + MV^2$ (3)

Subtracting L times (1) from (2) gives $MVL = Iw \rightarrow \frac{M^2V^2L^2}{I} = Iw^2$ which we can substitute into (3) to get

$mv^2 = m(v - \frac{M}{m}V)^2 + MV^2(1 + \frac{ML^2}{I})$

$= mv^2 - 2MvV + \frac{M^2V^2}{m} + MV^2(1 + \frac{ML^2}{I})$

$\rightarrow 2v = V(\frac{M}{m} + 1 + \frac{ML^2}{I}) = V(\frac{M}{m} + 5)$

$$\boxed{V = \frac{2mv}{M+5m}}$$

Substituting into (1) gives $\boxed{v' = \frac{(5m-M)v}{M+5m}}$

Substituting into (2) gives $w = \frac{mL}{I}(v - v') = \boxed{\frac{mvL}{I}(\frac{2M}{M+5m})}$

b. In order for a second collision to occur, we need to consider v' relative to V (i.e. consider the motion of the point mass in the frame of the triangle). Clearly if $v' > V$, there will be no second collision since the point mass will 'escape' the triangle's reach - to see this clearly, draw the circle centered at the triangle's center (i.e. the circumcircle of the triangle). The only possible point of contact for the second collision is at the bottom of this circle so the point mass must remain stationary relative to the triangle so that the triangle can rotate and collide with the point mass a second time.

Similarly, if $v' < V$, then there will be no second collision.

If $v' = V$, then indeed there will be a second collision since this means the point mass remains stationary relative to the triangle while the triangle will spin with angular velocity $w > 0$.

Thus, a second collision occurs iff $V = v'$ or

$2m = 5m - M$

$\boxed{M = 3m}$

c. Note that the three conservation laws still hold for the second collision. It is easy to observe that $\boxed{v'' = v, w' = 0, V' = 0}$, i.e. the point mass continues with its original velocity and the triangle comes to rest.

3 Problem 3

a. Because all projectile motion is parabolic, and parabolas are symmetric with respect to their vertex,

$$x = d$$

In order to be the minimum energy, the projectile must just pass the corners of the wall, giving the following equations

$$x_f = vcos(\theta)t_1 = d - l/2$$

$$y_f = vsin(\theta)t_1 + 1/2gt_1^2 = 2l$$

$$x_f = vcos(\theta)t_2 = d + l/2$$

$$y_f = vsin(\theta)t_2 + 1/2gt_2^2 = 2l$$

substitution of t_1 yields the following value for v^2:

$$v^2 = \frac{g(d+l/2)^2}{2cos^2(\theta)(2l - tan(\theta)(d+l/2))}$$

substitution of t_2 and v^2 gives the following expresion.

$$2l = tan(\theta)(d - l/2) + \frac{(d-l/2)^2}{(d+l/2)^2}(2l - tan(\theta)(d+l/2))$$

Solving for $tan(\theta)$ gives

$$tan(\theta) = 2l\frac{(d+l/2)^2-(d-l/2)^2}{(d+l/2)(d-l/2)l} = \frac{4ld}{(d+l/2)(d-l/2)}$$

which gives a solution for θ in terms of the given variables. Plug this value of θ into above equation for v^2 to get solution for v.

b. Solution not provided

4 Problem 4

a. First consider the frame of reference moving with the incline. Choose a coordinate system such that the positive x axis points toward the top of the incline and y is normal to the incline. Using Newton's second law gives

$$y:\ N - mg\cos\theta - mA\sin\theta = 0$$

$$x:\ N - mg\cos\theta - mA\sin\theta = F_m$$

We have taken the fictitious force in the reference frame into account (A is the horizontal acceleration of the incline). Back in the lab frame, the incline experiences forces that can be calculated from Newton's third law. In the horizontal direction,

$$MA = \mu N\cos\theta - N\sin\theta$$

Let $k = m/M$. Then

$$N = \frac{mg\cos\theta}{1-k\mu\sin\theta\cos\theta + k\sin^2\theta}$$

and

$$F_m = -mg\sin\theta + N(\mu + k\mu\cos^2\theta - k\sin\theta\cos\theta)$$

so

$$F_x = -mg\sin\theta + mg\cos\theta\left(\frac{\mu + k\mu\cos^2\theta - k\sin\theta\cos\theta}{1-k\mu\sin\theta\cos\theta + k\sin^2\theta}\right)$$

The net torque on the sphere is

$$\tau = -\mu NR = -\mu R\frac{mg\cos\theta}{1-k\mu\sin\theta\cos\theta + k\sin^2\theta}$$

Given that the ball travels $h/\sin\theta$ to the top of the incline and rotates through an angle $h/R\sin\theta$,

$$v_m = \sqrt{-2gh + 2g\cot\theta\left(\frac{\mu + k\mu\cos^2\theta - k\sin\theta\cos\theta}{1-k\mu\sin\theta\cos\theta + k\sin^2\theta}\right)}$$

$$\omega_m = \sqrt{\omega_0^2 - \frac{5gh\cot\theta/R^2}{1-k\mu\sin\theta\cos\theta + k\sin^2\theta}}$$

Using conservation of momentum,

$$v_M = -k\sqrt{-2gh + 2g\cot\theta\left(\frac{\mu + k\mu\cos^2\theta - k\sin\theta\cos\theta}{1-k\mu\sin\theta\cos\theta + k\sin^2\theta}\right)}$$

b. The change in energy (which can be calculated) is, in the moving frame:

$$\Delta E = \frac{1}{2}mv_m^2 + \frac{1}{2}I\omega_m^2 + mgh - \frac{1}{2}I\omega_0^2$$

c. Solution not provided