# 2013 WMI Mini Lecture Outlines

# 2013 WMI Mini Lecture
# Mathematical principles within famous painting

**DAHEE CHUNG, CSIA (1997dahee@gmail.com)**

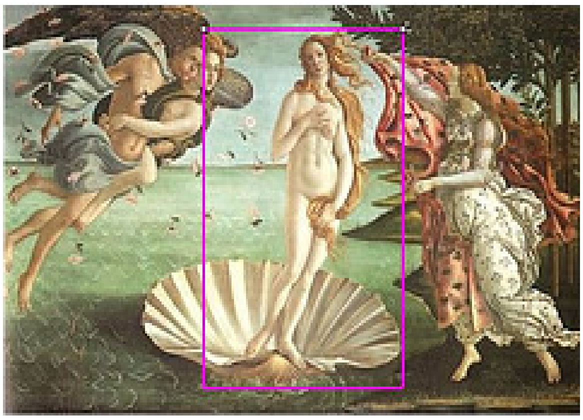
**August 15, 2013**

## 1. Introduction

Welcome! It's my pleasure to give you a lecture in this World Mathematics Invitational! The goal of my lecture is to appreciate masterpieces and understand mathematical principles within each of pictures. We are going to skim masterpieces of various painters and catch mathematical tools and secrets from the pictures. After the brief explanation about famous paintings, we are going to understand mathematical meaning within the masterpiece and solve related problems. Aren't you interested? Let's start!

## 2. What is the golden ratio found in the body of *'The birth of Venus'* and *'Hector and Andromache'*?

The golden ratio (symbol is the Greek letter phi) is a special number approximately equal to 1.618. It appears many times in geometry, art, architecture and other areas. If you divide a line into two parts so that: the longer part divided by the smaller part is also equal to the whole length divided by the longer part. Then you will have the golden ratio.

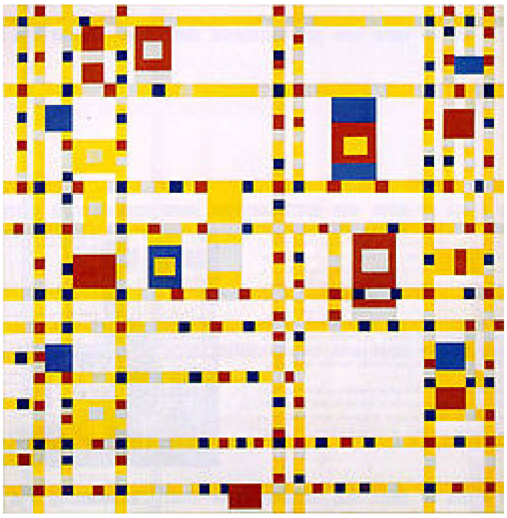


***Related concepts***

♠ How many golden rectangles are in *'Broadway Boogie-Woogie'*?

This rectangle has been made using the Golden Ratio, Looks like a typical frame for a painting, doesn't it?

Some artists and architects believe the Golden Ratio makes the most pleasing and beautiful shape.



Here is one way to draw a rectangle with the Golden Ratio:

- Draw a square (of size "1")
- Place a dot half way along one side
- Draw a line from that point to an opposite corner (it will be $\sqrt{5}/2$ in length)
- Turn that line so that it runs along the square's side

Then you can extend the square to be a rectangle with the Golden Ratio.

♠ <u>A special relationship between the Golden Ratio and the Fibonacci Sequence</u>

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ... The next number is found by adding up the two numbers before it. And here is a surprise: if you take any two successive *(one after the other)* Fibonacci Numbers, **their ratio is very close to the Golden Ratio**.

In fact, the bigger the pair of Fibonacci Numbers, the closer the approximation. Let us try a few:

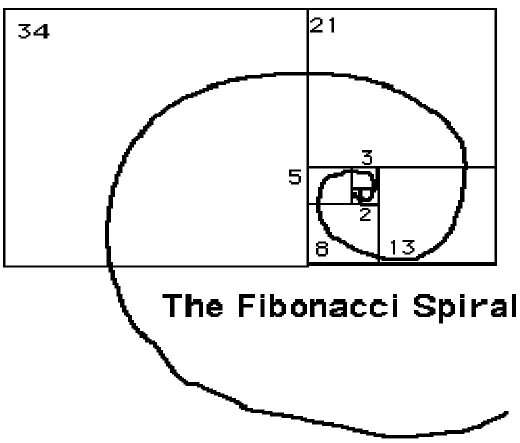
| A | B | B/A |
|---|---|---|
| 2 | 3 | 1.5 |
| 3 | 5 | 1.666666666... |
| 5 | 8 | 1.6 |
| 8 | 13 | 1.625 |
| ... | ... | ... |
| 144 | 233 | 1.618055556... |
| 233 | 377 | 1.618025751... |
| ... | ... | ... |

## 3. Can we draw the spirals in *'The shell'* and *'Tower of Babel'* with mathematical methods?


In mathematics, a **spiral** is a curve which emanates from a central point, getting progressively farther away as it revolves around the point.

Two major definitions of "spiral" in a respected American dictionary are

**a.** A curve on a plane that winds around a fixed center point at a continuously increasing or decreasing distance from the point.
**b.** A three-dimensional curve that turns around an axis at a constant or continuously varying distance while moving parallel to the axis; a helix.
We can build a squarish sort of nautilus by starting with a square of size 1 and successively building on new rooms whose sizes correspond to the Fibonacci sequence:




Running through the centers of the squares in order with a smooth curve we obtain the nautilus spiral , the sunflower spiral. This is a special spiral, a self-similar curve which keeps its shape at all scales (if you imagine it spiraling out forever).

It is called equiangular because a radial line from the center makes always the same angle to the curve. This curve was known to Archimedes of ancient Greece, the greatest geometer of ancient times, and maybe of all time.

## 4. What is the secret of magic square hidden in '*Melencolia I*' and '*Ssireum*'?

A **magic square** is a square array of numbers consisting of the distinct positive integers 1, 2, ..., $n^2$ arranged such that the sum of the $n$ numbers in any horizontal, vertical, or *main* diagonal line is always the same number, known as magic constant.

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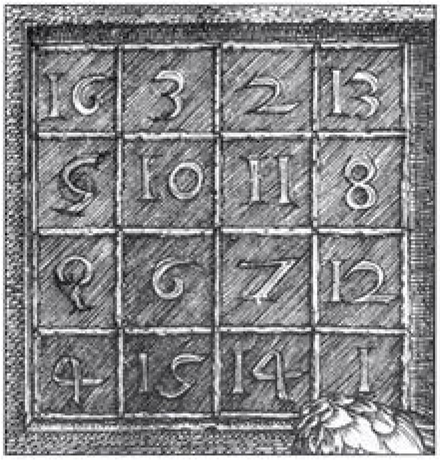
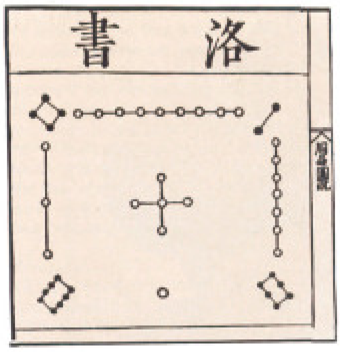
The 4 × 4 magic square, with the two middle cells of the bottom row gives the date of the engraving: 1514. This 4x4 magic square, as well as having traditional magic square rules, its four quadrants, corners and centers equal the same number, 34, which happens to belong to the Fibonacci sequence. His age in 1514 was 43, reverse of 34.




| | | |
|---|---|---|
| 8 | | 5 |
| | 2 | |
| 5 | | 2 |




| | | |
|---|---|---|
| 4 | 9 | 2 |
| 3 | 5 | 7 |
| 8 | 1 | 6 |

## 5. Further Discussions

-What is the probability that swindler will win in ‘*The cheat with ace of diamond’*?
-What is the hidden meaning of pentagon and star in *‘The Descent from the Cross’*?
-Where is the center of gravity in *'Woman holding a balance'*?
-What are secrets of *'The school of Athens'* and *'Brena Madonna'* which are drawn with the utilization of perspective?
-What symbolize the cabbalistic number and sign in *'Last Supper'*?
- What is volume relationship between the figures in ‘*Senecio*’
-Why the impossible triangle in '*Waterfall*' seems possible?
-What is the principle of melodious note in *'Spanish singer'* and *'Three musicians'*?
-*'The red son gnaws at the spider'* and dancing pi?
-What is the balance between circles within the *'Several Circles'*?
-What is the meaning of clock in ‘*The Persistence of Memory’*?
-Why the legs of easel in ‘*The Human Condition’* and ‘*atelier*’ are three?

*If you have any questions, please contact me by 1997dahee@gmail.com.*

# 2013 WMI Mini Lecture
# Living Amid Probabilities

**Joon-hyuk Chang, KMLA(edwardjh@naver.com)**
**Minseok Kim, KMLA (aosdbqhdl@naver.com)**

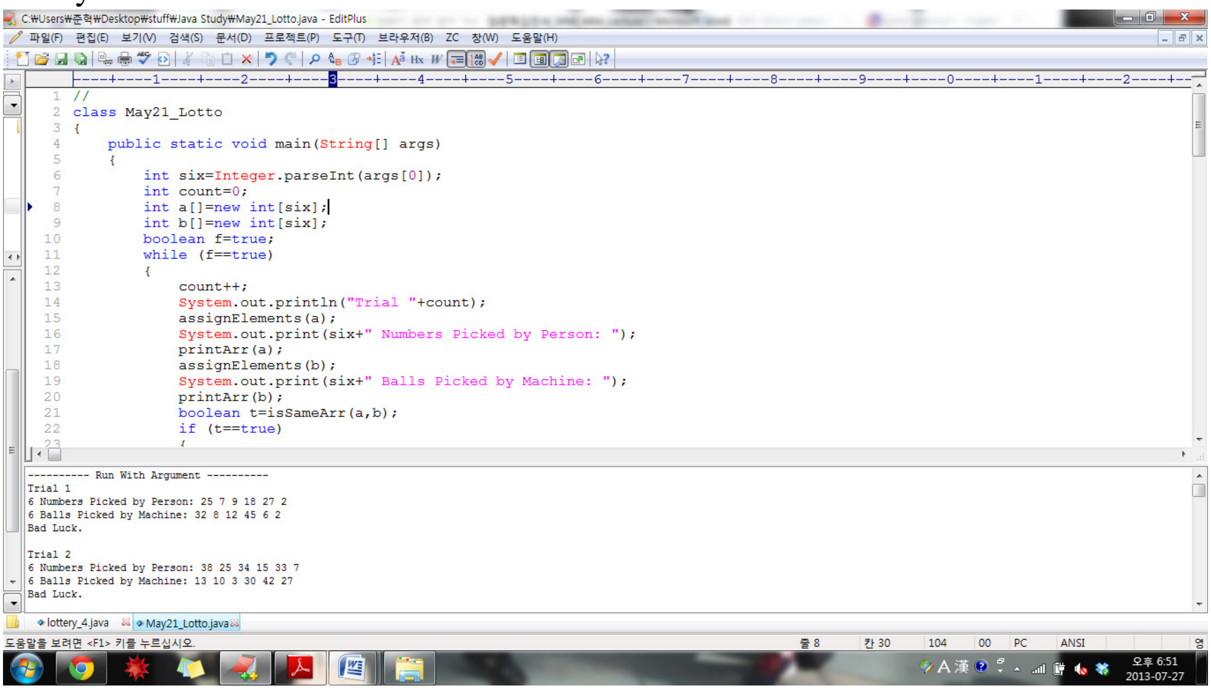
**August 15, 2013**

## 1. Introduction

Greetings to all. It is with the utmost pleasure that we address you. Today, we will deliver a lecture on probabilities in our everyday lives. We have one small hope. We wish that this lecture will be memorable to most, if not all, of you, a unique lecture that many of you find striking. We will address topics such as lotteries, dice, and card games one at a time and undertake an examination at the probabilities involved in each of these issues. Sit back, relax, and enjoy our lecture. Let's begin.

## 2. Lotteries – Part 1

How much Lotto do you think you have to purchase before you win the first prize? Can you try to make a conjecture?
In case some of you are not familiar with how lotteries work, I would like to explain. When you buy a lottery, there are six numbers imprinted on it. The six winning numbers, which are announced periodically, are chosen randomly one at a time. How many same numbers you have determines the prize you will receive. If all the winning numbers are imprinted on the lottery you bought, congratulations: you have won the lottery.

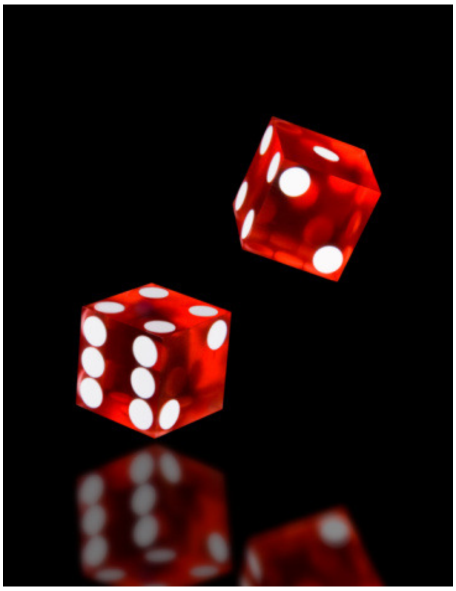


Now, I would like to introduce you to my self-made lottery program.
Let's define a *trial* as the process of buying a lottery and checking the winning numbers. When I enter an integer between 1 and 6, inclusive, it runs a random simulation of consuming lotteries and counts the number of trials before there are as many matches between the winning numbers and the purchased lottery's numbers as the number I put as input. For example, if I enter 1, it shows me how many trials of buying lotteries I would have to undergo until one out of the six numbers match. The same goes with 2, 3, 4, 5, and 6.
How much time do you think will be needed for the program to indicate that I won first prize in the lottery? Let's begin running the program now with 6 as the input and continue discussing lottery more in depth near the end of the lecture.

## 3. Dice

The six-sided dice has many properties that make it intimate with concepts of probability. Each of the six sides has distinct whole numbers from one to six inscribed on it. The theoretical probability of attaining each number when throwing a regular dice is $1/6$.
*__Theoretical Probability__: the probability we would expect to observe
*__Experimental (Empirical) Probability__: the probability actually observed during testing processes

Understanding the distinction between theoretical probability and experimental probability is important. Throwing a large number of dice or throwing one dice multiple times illustrates the difference clearly. If, for instance, 12 dice are thrown, we would expect to get each of the six numbers two times. That is based upon theoretical probability. However, when we actually throw 12 dice, we can easily see that it is actually difficult to get each number exactly two times. Most likely, we are going to get some numbers more than others. The uneven distribution of the attained numbers can be easily noticed if we create six stacks of dice with each stack containing dice with the same number. After throwing the dice, if we calculate the probability that each number was attained, we will end up with the experimental probability.



An interesting probability I would like to point out is that the more dice rolled, the closer the experimental probability will be to the theoretical probability. For example, when we roll 36 dice and stack them into groups of same numbers again, we can notice that the lines are much more even then before. We receive a result much closer to what we can expect before actually throwing the dice.

## 4. Card Games

Cards are also items closely related to concepts of probability.
A typical deck of cards contain 52 cards, not counting the usually included two Jokers. In a deck, there are four suits – clover, spade, diamond, and heart – and thirteen ranks – numbers one through ten, Jack, Queen, and King. All of the cards are distinct, which means that although two different cards from the same deck may have the same suit or the same rank, they may not have the same suit and the same rank simultaneously.
Although all card games essentially and inadvertently involve probability to some degree, the two to discuss in this lecture are Poker and Blackjack.

1. Poker

Poker is a common gambling game involving betting. The person who can create the best combination of cards using his own cards and the cards on the table at the end of the game is declared the winner of a round of Poker.
When the game begins, the dealer gives two cards to each of the players. The cards that a player holds are called his 'hand.' Afterwards, three cards on the top of the deck are flipped so that every player can see them. The dealer flips another card, and the players take turns betting. This process is repeated until there is a certain amount of cards on the table. The players who had not folded reveal their hands, and the winner takes the pot.
The aspect of poker that makes it especially intimate with probability is the ranking order of the hands. The hierarchy is arranged in the order of the least likely outcome to the outcome with the highest probability.
The aspect of poker that makes it especially intimate with probability is the ranking order of the hands.

| Hand | Frequency | Probability | Odds |
|---|---|---|---|
| Royal Flush | 4 | 0.000154% | 649,739:1 |
| Straight Flush | 36 | 0.00139% | 72,192.33:1 |
| Four of a Kind | 624 | 0.0240% | 4,165:1 |
| Full House | 3,744 | 0.144% | 694.17:1 |
| Flush | 5,108 | 0.197% | 508.80:1 |
| Straight | 10,200 | 0.392% | 255.8:1 |
| Three of a Kind | 54,912 | 2.11% | 47.33:1 |
| Two Pair | 123,552 | 4.75% | 21.04:1 |
| One Pair | 1,098,240 | 42.3% | 2.37:1 |
| High Card | 1,302,540 | 50.1% | 1.96:1 |

2. Blackjack


Blackjack is another well-known card game with an exceptional proximity to probability concepts.

The goal of Blackjack is to reach near as possible to 21. Each of the cards has an assigned value. Cards with ranks two to ten each has a value same as its rank. Jack, Queen, and King have values of ten. Ace can represent a value of either one or eleven, depending on how the player wants to use it.

The dealer deals each player two cards – one face-up, one face-down. Each player take turns deciding whether to receive more cards (also known as 'hit') or stop. The moment a player's hand exceeds 21, he is 'bust,' meaning that he is automatically out of the game. When all players have stopped, they reveal their hands, and the winner is the person whose hand's value is the closest to 21.

Blackjack is a game where the odds of any particular outcome can be calculated extremely accurately. By looking at the face-up cards of other players and your own cards, you can guess what cards are remaining in the deck. Thus, you can determine the approximate probability that you will not be bust when you hit. If there is a fair chance that you will get closer to 21 without exceeding it, it would be rational to hit. One must remember that there is a greater probability of being dealt a card of value 10 than cards of any other values.

It is interesting to note that the probability of being dealt a 21 – a Blackjack – is 4.83%. In a deck of 52 cards, there are four Aces and 16 cards with values of ten. Since you need one Ace and one card of value ten to get 21, there are $4*16=64$

possibilities. The total number of outcomes can be calculated as corresponds to $_{52}C_2 = 1326$ . Thus, the probability of being dealt a Blackjack is $64/1326 = 0.0483 = 4.83\%$. This particular percentage means that you will be dealt a Blackjack in about once every 21 rounds.

## Lottery-Part 2

The probabilities of having a particular number of matches are as below.

| Number of Matches | Probability | Calculations |
|---|---|---|
| 6 | 1/8145060 | $\frac{1}{_{45}C_6} = 1\Big/\left(\frac{45*44*43*42*41*40}{6*5*4*3*2*1}\right)$ |
| 5 | 1/34807.9487 | $\frac{_6C_5 * _{39}C_1}{_{45}C_6}$ |
| 4 | 1/732.7989 | $\frac{_6C_4 * _{39}C_2}{_{45}C_6}$ |

Thus, in order to win the first prize in the lottery, you would have to beat the one-in-8145060 probability.
You can notice that the Java lottery program is still running. At the rate it is operating, it will take about one day and a half before it wins the lottery.

This is the end of the lecture. Thank you.

# 2013 WMI Mini Lecture
# Inequalities

**In Ha Lee, HAFS (inha812@gmail.com)**

**August 15, 2013**

## 1. Introduction

Hello and welcome to the 2013 World Mathematics Invitational! I am In Ha Lee, and I will give a brief lecture on math inside economics. I have chosen this topic because I don't think you guys know much about this topic.
Before all this, why is math inside economics a good topic? Economics and mathematics love each other. No, let me correct it. Economics love math. Pretty much everything about economics is math. In the next 15 minutes or so, I will show you why pretty much all of economy has something to do with math.

## 2. Elasticity

First, let's look into something called elasticity. In economics, elasticity is the measurement of how changing one economic variable affects others.
For example:

"If I lower the price of my product, how much more will I sell?"
"If I raise the price, how much less will I sell?"
"If we learn that a resource is becoming scarce, will people scramble to acquire it?"

Generally, an elastic variable is one which responds a lot to small changes in other parameters. Similarly, an inelastic variable describes one which does not change much in response to changes in other parameters.
In more technical terms, it is the ratio of the percentage change in one variable to the percentage change in another variable. It is a tool for measuring the responsiveness of a function to changes in parameters in a unit less way.
The definition of elasticity is based on the mathematical notion of point elasticity. In general, the "*x*-elasticity of *y*", also called the "elasticity of *y* with respect to *x*", is:

$$E_{y,x} = \left|\frac{\partial \ln y}{\partial \ln x}\right| = \left|\frac{\partial y}{\partial x} \times \frac{x}{y}\right| \approx \left|\frac{\%\Delta\Delta}{\%\Delta\Delta}\right|$$

The approximation becomes exact in the limit as the changes become infinitesimal in size. The absolute value operator is for simplicity – generally, depending on context, the sign of the elasticity is understood as being always positive or always negative. However, sometimes the elasticity is defined without the absolute value operator, when the sign may be either positive or negative or may change signs. A context where this use of a signed elasticity is necessary for clarity is the cross-price elasticity of demand — the responsiveness of the demand for one product to changes in the price of another product; since the products may be either substitutes or complements, this elasticity could be positive or negative.

## 3. Game Theory

Next is about game theory. As some of you may know, game theory is related to real life more than other theories do. Before we go on, what is game theory?
Game theory is a field in applied mathematics basically focusing its studies in strategic decision making.
Some people call this the interactive decision theory, because it mainly is about making your own decision based on other people's already-made decisions. Game theory is used in various fields of study such as economics, political science, psychology, and even biology, so learning game theory enables you to access other fields of study.
Up to this point, you might think that there isn't much math in this field. I thought so too when I first met game theory. I'll show you the fundamentals of this theory and you'll understand why this is math.

## 4. Nash Equilibrium

Everything starts with the Nash Equilibrium. The Nash Equilibrium is a solution concept on a non-cooperative game involving two or more players. Non-cooperative literally means, the players make their decisions independently. In a Nash Equilibrium state, each player is assumed to know the equilibrium strategies of the other players, and no player has anything to gain by changing only their own strategy unilaterally. You might not understand what this means, since they are all words.

| | | B | |
|---|---|---|---|
| | | Confess | ZIP |
| A | Confess | (8, 8) | (1, 10) |
| | ZIP | (10, 1) | (4, 4) |

This is an example of the Nash equilibrium. You guys all heard of the prisoner's dilemma, right? That's also Nash Equilibrium. Say A and B represents prisoners in jail, and they have two choices: they can either confess or keep their mouth shut. The numbers above are the amount of years sentenced to the following person, A and B respectively. Now let's compare each outcome. Let's compare them side by side, and up and down. You can see that for both of them, confessing gives better results than

keeping their mouth shut. Therefore, as rational decision makers, they inevitably start talking even though the best option for both of them is to keep their mouth shut.
This tells us, that the best choice may not always lead to the best outcome.

## 5. Conclusion

Now, why is this an important lecture? Mathematics for mathematics itself is cool indeed. However, I think math comes to importance when it solves real life problems. Mathematics is the basics of economics, engineering, various fields of natural science, psychology, statistics, etc. Studying mathematics will give each and every one of you an advantage in these fields when you grow up. Now, you may all be planning to major in math, but I present you with this lecture, pointing out that many other things can be solved with math, and they are just as fun as math is. I want all of you to think about this matter and hope that I gave you guys some help in figuring out the essentials of economy through math.

# 2013 WMI Mini Lecture
# Probability Soccer

**Sang Jun Han, CSIA (hansang0282@gmail.com)**

**August 15th, 2013**

## 1. Introduction

Hello ladies and gentlemen, I wish you are having a great time!
The lecture that I would like to present to you today is of very, very simple everyday math, but math that so much people even bet their money on. You can't see it. As we couldn't know about the microorganisms until we invented the microscope, this math is secretly dissolved in every aspect of our life. If you get a keen eye looking into them, you would be able to see math lingering around just like the microbes. Today, I am here to be that eye for you and look into the math of soccer, specifically the most heart-beating part of it.

## 2. Soccer: The Penalty Shootout

Soccer is an all-around sport, which you can play if you only have any type of a ball that is kickable. You score if you put the ball into the other team's net. You win if you score more than the other team. To score, you can do whatever you want except using arms and hands and keeping the offside rules. I can daresay that soccer is one of the ball games with the simplest rules.
Even though the time in which you get pressure and tension could be many, such as when the striker is about to score the game-ending goal, or when Cristiano Ronaldo is about to take a shot of a free kick, I would like to tell you about one moment in which you would get the most tension: the penalty shootout. It is the Ball version of the Russian roulette. You take a shot at the goalie just like the situation in which you get the penalty award. Then, the other shoots at your team. The fact is, when you fail it, you fail the whole team. The saver becomes so glad, but the kicker feels as if the world is falling upon them. It is simply hard to explain the extreme contrast between the two players.
What we would do together for today is digging up the behind story of penalty shootouts, and finding out whether as a mathematician can predict the result of it.

## 3. Keeper's Dilemma

Starting with, we got to realize about how the Penalty Shootout is done. The fight between the keeper and the kicker is only 11 meters apart. The kicker has a freedom to choose where to start his running, but the keeper must stay within the goal line in

between the goal posts until the kicker touches the ball. The kicker must touch the ball without stopping, and after he touches the ball in whatever means, he cannot touch it again. Evaluating from these rules, there is much more restriction to the keeper than the kicker.
So when the best kickers, Cristiano Ronaldo, Lionel Messi, Wayne Rooney, kicks their best, the ball in average reaches up to 80 kilometers per hour. Considering that the distance between the keeper and the kicker is 11 meters, we can do a simple calculation to find out that the ball reaches the keeper in no less than 0.5 seconds.

The speed of the ball: $80\,\text{km/h} \times 1000\,\text{m/km} \times 1\,\text{hour/3600sec} = 22.22\,\text{m/s}$
The time the ball takes to reach the keeper: $11\text{m} \div 22.22\,\text{m/sec} = 0.495\text{sec}$

But the experiments with the best savers, say Iker Casillas, Gianluigi Buffon, Petr Cech, announces the fact that the time it took for the keepers to react after realizing where the ball would go was at best 0.75 seconds on the best conditions. Definitely no keepers can save the penalty if they react after they realize where the kicker shot the ball. How do we then see keepers blocking the penalty? Yes, as you might have figured, they guess. They jump right away when the ball is about to be kicked. Then they wish for the goddess of victory to take their hands up. People who watch soccer sometimes refer to this situation in which there is absolutely no way for the keeper to certainly stop the ball as, "Keeper's Dilemma".

## 4. Simple Possibility

We have found out that for the keeper, it is more of a guessing game than a playing game. Then we might ask about a simple possibility for the keeper to save the ball by luckily having the kicker kick at the place the keeper jumped. Do not forget that we are trying to get the simplest possibility possible, which, in most of the time, gives us the most correct answer.
Thinking about the possibility of stopping the ball, we can measure the total area of the goal post to the total area of the goalie. A normal goal post has a length of 24 feet and width of 8 feet. A simple multiplication gives us, $24 \times 8 = 192$ feet squared. This equals to 17.8608 square meters. Now, a normal keeper in the soccer league is about 1.9 meters tall, and his shoulder width is about 0.7 meters. Again, we will do a simple multiplication to get $1.9 \times 0.7 = 1.33$ meters squared. So, the percentage of the area of the goal post which is covered by the goalie is,

$$\frac{\text{Area of the Keep}}{\text{Area of the Goal Post}} = \frac{1.33\text{m}^2}{17.8608\text{m}^2} = 7.44\text{percent}$$

In other words, a keeper has a fat chance of 7.5 percent to be able to stop the ball. Considering the fact that it is only this big of a percent that the keeper hopes for and throws his comparatively teeny body even arouses some sympathy for the keeper.

# 5. It is Not THAT Simple

Don’t worry. The game of soccer isn’t that unfair. The fact is, not a lot of kickers have enough guts to actually kick into the middle. Yes there are literally countable people, such as Antonin Panenka a Czech footballer who surprised the world by ‘floating’ the ball so slowly and directly into the middle of the goalpost. The kick in which you float the ball right into the middle of the goalpost from then was called as the Panenka Kick. Most recently, Andrea Pirlo did it on Euro 2012, and I couldn’t find anymore. Yes, unless the kicker makes a mistake during his kick, the statistics say that it is very, very unlikely for the kicker to kick the ball in the middle.
Aha, a bit of a relief. Then how shall we now divide the goalpost after knowing this fact? Here, we need to specifically define the area in which the goalie can defend just standing at his starting point. Leonardo da Vinci’s drawing of a man gives us the fact that a man’s arm length equals to that man’s height. Also, the kicker wouldn’t probably wonder whether to kick the upper middle or the down middle. So, we can delete a rectangular area of the goal for the goalie. The area in which the keeper can defend while simply standing can be calculated as,

$$\text{(The Arm Length)} \times \text{(The Goal Post's Height)} = 1.9 \times 2.4384 = 4.63206\ \text{m}^2$$

This would definitely reduce the area in which the keeper should take care of by the amount of about a quarter. Now, if we consider the area that the keeper can save if ball comes as circular, we can calculate its size by defining the arm’s length as the diameter. Then, we can get the value of the area of the circular part which is,

$$\pi\left(\frac{d}{2}\right)^2 = 3.14 \times \left(\frac{1.9}{2}\right)^2 \approx 2.83\text{m}^2$$

Here, we would give a difference in thought. We will now say that the keeper would, and is also done by the keepers normally, decide to jump in either four corners of the goalpost. In other words, the keeper would jump to the left up, left down, right up, and right down. For this, we can simply say that the keeper would be able to jump to each of the corners in 25 percent chance each. Now, we should get the area of each of the corners. The area of each of the corners can be easily derived by dividing the remaining area of the goal by 4. The result is, $(17.8608 - 4.63296) \div 4 = 3.30696\text{m}^2$.
The chance of the keeper saving is done by the given thought process.

$$\text{(The chance the keeper would choose the correct corner)} \times \text{(The chance the keeper would save when he chose the correct corner)} \times 100 = 1/4 \times 2.83/3.30696 \times 100 = 21.39\text{percent}$$

A notable fact from this thought process is that if the keeper chooses the correct space to jump for, then the percentage of him blocking that area is,
$(2.83/3.30696) \times 100 = 85.57\text{percent}$

## 6. Conclusion

Yes, it is true that there are many ways to approach to this question. I would say that no approach to this question is wrong. However different the chance may come out, it still isn't reassuring to the keeper because it would never go over 50% unless you decide to say that the keeper would block it or not, and give 50% chance.
To be more exact about this situation, the fact is that all keepers are aware of the style of the kicker. Before standing before the ball, they are always informed about how the kickers mostly kick when they are in a burden to score. Even more, the penalty shootout is done when the athletes have already run 120 minutes. Any type of a tendency to kick is most likely to show up. The keeper, who is mostly the person with the most stamina left, can catch that glitch of moment. This is why there seems to be so much super saves that just cannot be explained by the percentage that we would get mathematically.

# 2013 WMI Mini Lecture
# Relationship between Music and Mathematics

**Da Yeon Lee, CSIA (leedygood7@gmail.com)**

**August 15th, 2013**

## 1. Introduction

Good afternoon everyone! This is Da Yeon Lee from CSIA. Today I am very honored to have an opportunity to talk in front of all of you. The topic for today's talk would be, as you guys all know already, the mathematics of the music. When I was young, I learned how to play clarinet. You know, when you do music, many people say you have to "feel the music by heart" and such things. Music is generally taken as something very emotional. And the image of mathematics was rather the very opposite of it – rational and very logical. I thought they were two separate things. But as I grew older, I was pretty astonished by the fact that mathematics was also working as the base of these musical works. Now I'm going to talk about some mathematical factors in the music, and I hope all of you enjoy this talk.

## 2. Pythagoras

Surprisingly, Pythagoras was among the first ones to construct a theory related to music. (especially the pitch of the sound) According to the record by Boethius, a philosopher from 6th century, Pythagoras was walking by the blacksmith's workshop when he heard the perfectly harmonious sounds of the hammering sounds. Later he found out that the ratio among the weight of the hammers was 6:8:9:12, integer ratio. Especially when the weights were 2:1, they made sounds that had a difference of an octave. When the weight ratio was 3:2, they made perfect 5th (diapente) sounds; when the weight ratio was 4:3, they made perfect 4th (diatessaron) sounds.
This result is quite hard to believe, especially since many of the researchers had already tried making sounds with the hammer of specific ratio, but did not make such sounds. Thus this tale may be legendary – only a myth – but still this is significant in that Pythagoras related numbers and ratios with music. And these ratios can actually be applied in case of the strings. While the ratios did not work out in the case of weights of the anvils, if we apply those ratios in the lengths of the strings, then we can find out that the harmonies that Pythagoras talked of works.
When the length of the string is long, it makes lower sound; when the length is short, it makes higher sound. And amazingly, the ratios of the length introduced above (2:1, 3:2, 4:3) show the same relationship with harmonies of the sound.

## 3. Harmonic Sequences in an Octave

One octave is consisted of 12 different sounds. And among the designation of the tones, arithmetic and harmonic sequences are used.
There are two ways to calculate the frequency of the semitones: we may use integer ratios as presented above by Pythagoras, or use the temperament. Most commonly used system of the tunes is "equal temperament": as expressed in keyboard instruments, one octave is made up of 12 different tones, divided by semitones. The semitones show that respective frequency ratio is constant. According to this, each semitone must have the frequency ratio of $12\sqrt{2} \approx 1.0595$.

| | Integer ratio | temperament |
|---|---|---|
| Perfect 4th | $\frac{4}{3} \approx 1.3333$ | $(12\sqrt{2})^5 \approx 1.3348$ |
| Perfect 5th | $\frac{3}{2} \approx 1.5$ | $(12\sqrt{2})^7 \approx 1.4983$ |

By the above example, which represents the chords in both integer ratio and temperament, we can see that both systems show similar results in calculating the difference. Such method of calculation used in temperament shows the usage of harmonic sequence in music.

## 4. Mersenne's Laws

Marin Mersenne was a French theologian, philosopher, mathematician and music theorist in 17th century. He is often referred to as the "father of acoustics." He is well known for his works on relating music and sound with mathematical concepts, now known as Mersenne's Laws.
*L' Harmonie Universelle* (1637) contains the idea of Mersenne's Laws, which describe the frequency of oscillation of a stretched string. The main ideas of the Mersenne's Laws are:

1. The frequency is inversely proportional to the length of the string (this is usually credited to Pythagoras himself)
2. The frequency is proportional to the square root of the stretching force
3. The frequency is inversely proportional to the square root of the mass per unit length.

Thus the formula for the lowest frequency is

$$f = \frac{1}{2L}\sqrt{\frac{F}{\mu}}$$

where $f$ is the frequency, $L$ is the length, $F$ is the force and $\mu$ is the mass per unit

length.
I am introducing this formula just to show the relationship of each factor to the frequency of the sound, so you don’t really have to memorize or have detailed knowledge about the formula right now. (If you are interested, you can certainly do some research and study on acoustics!) When we apply such relationship (inversely proportional / proportional factors) into the string instruments, let’s say, for example, a guitar, then what can we do to make a sound with higher frequency? First, you can shorten the length of the vibrating string by using a capo for guitar. Second, you can increase the tension of the string by pulling it with stronger force. And lastly, you can use thinner, lighter string to make higher sounds.

## 5. Conclusion

So now, we have taken a look at how mathematics is deeply affecting the music as well. Mathematics does really seem to be around us, doesn’t it? It was a short talk and I may still have insufficient knowledge on the subject. Still, it was nice to have an opportunity to talk to you about my interests and how I found such subject fascinating. Hope you all enjoyed as well, and thank you for listening!

# 2013 WMI Mini Lecture
# Size of Infinite Sets

**Jae Hee Lee, CSIA** (skan9015@gmail.com)

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## 1 Introduction

Welcome to 2013 World Mathematics Invitational!

In this mini-lecture, I would like to present how to study the size of infinite sets. What does it mean for an infinite set to have a size? How is it related to the usual notion of the size of finite sets? How do we compare the size of infinite sets? How big are the sizes of well-known infinite sets such as the natural numbers(N), the integers(Z), the rationals(Q), the reals(R)? There is a huge theory called the set theory to systematically study these topics, but in this lecture I will focus more on the interesting facts that will surprise you rather than rigorously going through all the details of set theory. Let us start with the discussion of finite sets, which has an intuitive notion of size which we are already familiar with.

## 2 Sizes of Finite Sets

How do we know the size of finite sets? For finite sets, the definition of size captures the notion of the number of elements. If we denote the size of the set X as |X|, |X| simply counts the number of elements in the set X. For example,

$$
\begin{aligned}
|\{1, 2, 3, 4\}| &= 4 \\
|\{1, 8, 56, 947, 2013\}| &= 5 \\
|\{\text{dog}, \text{cat}, \text{elephant}\}| &= 3
\end{aligned}
$$

Therefore it is not difficult to think about the size of finite sets: you just count the number of elements in the set. It is also very easy to compare the size of two different sets. If two sets X and Y satisfy $|X| = m$ and $|Y| = n$ with $m < n$, then we could simply say that Y has more elements than X. Such an argument is quite valid and pretty much enough when we are dealing with finite sets. However, when we have to discuss infinite sets, the same approach works no more. What does it means to count the number of infinite sets? $N = \{1, 2, 3, \cdots\}$ obviously has no natural number corresponding to its size. Can we just say $|N| = \infty$? But we did not even define $\infty$. Even if we did, a new problem arises: how will we compare the sizes of infinite sets? Is $|N| = |\{2, 3, 4, \cdots\}| = \infty$? Is $|N| = |\{1, 4, 9, \cdots\}| = \infty$?Is $|N| = |\{2, 4, 6, \cdots\}| = \infty$? Do all infinite sets have the same size, namely $\infty$? To answer all these questions and to avoid contradictions, we have to examine the notion of the size of sets in a different way.

## 3 A New Way to Compare Sizes of Sets

Our new approach is to look at the functions from one set to another. Let X and Y be our sets, and see how functions can tell us about their relative sizes.
A function is a mapping from the elements of X to the elements of Y . That is, for each x 2 X, wend its unique match, partner, or value from Y . We call this match f(x), and it is one of the elementsy 2 Y . Saying that function f is from X into Y means that every element in X has a match in Y . Note that one element in X cannot have two matches, but the matches could be equal for dierent elements of X. Not all elements of Y have to be a match of some element of X. But if the two properties above actually do hold, then a function is special, and it reveals facts about the sizes of sets X and Y .

Let us use our examples above. Let A = {1, 2, 3, 4}, B = {1, 8, 56, 947, 2013}, C = {dog, cat, elephant}. Note that A, B, C are sets, not the sizes of the sets. Let's define a function f from A to B, or more concisely f : A → B, as follows.

$$
\begin{aligned}
f(1) &= 56 \\
f(2) &= 1 \\
f(3) &= 1 \\
f(4) &= 2013
\end{aligned}
$$

This is really a function because every element in A has found its match in B. The matches are not equal, but that is okay. But this function is not really interesting for our discussion of the size of sets. So let us modify this function a little bit. Let us say f(3) = 8 instead of f(3) = 1. Then different elements of A has different matches in B. Such functions are called one-to-one. Because for one-to-one functions we have to find a different match for every element in A, we could see that B has to have at least as many elements in A, or we will run out of diffe rent elements in B to be a match of elements in A.
This is true for any sets! So if there exists a one-to-one function from set X to Y , $|X| \leq |Y|$

This time let us look at the other property. Let us define a function g from A to C (g : A → C).

$$
\begin{aligned}
g(1) &= \text{cat} \\
g(2) &= \text{dog} \\
g(3) &= \text{cat} \\
g(4) &= \text{elephant}
\end{aligned}
$$

This is certainly a function. Also, for every element in C, there is an element in A that has its match as that element. 1 and 3 has match cat, 2 has match dog, 4 has match elephant. That covers all of the elements in C. A function satisfying such a property is called onto. Because for onto functions we have to find a match for every element in C, A should have at least as many elements in C, or we will run out of elements in A and some elements in C will not be matches of any elements in A. Remember that one element in A cannot have two matches, so for each element in C we need to find a different element in A.
This is also true for any sets! So if there exists a onto function from set X to Y , $|X| \geq |Y|$

Or we could just say this is our definition of ≤ and ≥ for size of sets. From now on, we do not count the elements in the sets we are interested in, but we just see if we could find a special function from one to another. Basically, what we are doing here is choosing one element from each set and match them, until one of the set runs out of elements.

Especially, if we could find a function that is both one-to-one and onto, this would imply $|X| \leq |Y|$ and $|X| \geq |Y|$. In other words, $|X| = |Y|$. Now we have built a new way to compare the sizes of sets, and we could expand this definition to all sets. We could now compare the size of infinite sets!

# 4 Comparing the Size of Sets

Now that we have the appropriate machinery to work on sizes of any sets, let's get to work!First of all, we could now precisely define finite sets as sets with size equal to $|\{1, 2, \cdots, n\}|$ for some natural number $n \in N$. We could also easily check that any finite set has smaller size than N, because we can never find an onto function from a finite set to an infinite set. We could also answer some of our original questions.

- Is $|N| = |\{2, 3, 4, \cdots\}|$? Yes, $f(n) = n + 1$ is both one-to-one and onto.
- Is $|N| = |\{1, 4, 9, \cdots\}|$? Yes, $f(n) = n^2$ is both one-to-one and onto.
- Is $|N| = |\{2, 4, 6, \cdots\}|$? Yes, $f(n) = 2n$ is both one-to-one and onto.

What? This is ridiculuous! What we have just shown is that N has the same size as its subsets that lacks some elements in the original set of natural numbers(proper subsets). The whole has the same size as its small part. This may be quite counter-intuitive, and even great mathematicians such as Gottfried Leibniz, the inventor of calculus, saw this fact as an outright contradiction. But using our new comparison standard, this turns out to be true. Infinite sets behave strangely.

In fact, even more unbelievable facts are waiting. We showed that some simple infinite subsets of N have the same size as N. It is not hard to show that any subset of N is either finite or has the same size as N. How about sets containing N?

In particular, let us look at the set of integers, $Z = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$. If we could find a one-to-one, onto function from N to Z, then $|N| = |Z|$. This set seems to have at least twice the elements of N and the chance of finding such function seems miserable. However, if we define $f : N \to Z$ as

$$\begin{aligned} f(1) &= 0 \\ f(2) &= 1 \\ f(3) &= -1 \\ f(4) &= 2 \\ f(5) &= -2 \\ &\cdots \end{aligned}$$

We could easily see that f is both one-to-one and onto. So $|N| = |Z|$.

But even more crazy results are waiting.
How about $Q = \{ p/q : p \in Z, q \in N\}$ Defining $f : N \to Q$ that is one-to-one and onto is more tricky. This time we order the rational numbers so that numbers using only 1 and −1 appears first, then the ones with 2 and −2, and so on. Of course we have to leave out the ones that appeared earlier. In other words, order the rationals in the following way:

$$\frac{0}{1}, \frac{1}{1}, \frac{-1}{1}, \frac{1}{2}, \frac{-1}{2}, \frac{2}{1}, \frac{-2}{1}, \frac{1}{3}, \frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}, \frac{3}{1}, \frac{3}{2}, \frac{-3}{1}, \frac{-3}{2}, \frac{1}{4}, \frac{3}{4}, \frac{-1}{4}, \frac{-3}{4}, \frac{4}{1}, \frac{4}{3}, \frac{-4}{1}, \frac{-4}{3}, \ldots$$

This ordering is itself a function from N to Q. It is certainly one-to-one. It is also not hard to see that all rational numbers will some time appear in this list, meaning that the function is onto. So $|N| = |Q|$.
If you think about it for a while, this is a very surprising fact. Clearly $N \subset Z \subset Q$, but $|N| = |Z| = |Q|$. This can never happen in finite sets, but with our new definition of comparing the size of sets allows us to derive such interesting results.
At this point, you may be curious if there are any infinite sets strictly greater than N in size. Sets as large as the Q that is large enough to densely occupy the whole real line has the same size with N. What can come next? Our following discussion is about sets larger than the set of natural numbers.

## 5 |R| > |N|

The real numbers is the set that has a strictly larger size than N. How do we show this? There is an obvious one-to-one function from N to R, namely $f(n) = n$.

So certainly $|N| \le |R|$. If we cannot find a function that is both one-to-one and onto, then this will give us the fact that the reals are larger than the naturals.
In fact, we can never find an onto function from N to R. So particularly there are no functions that are both one-to-one and onto. Let us assume that there is an onto function $f : N \to R$. If we could derive a contradiction from here, this will imply that our assumption was wrong, and therefore there are no such onto functions.
Since f is onto, each natural number has its real number match. This real number can be written in a decimal expansion: something like $3187.128398 \cdots$, 1.25765, $0.00012382 \cdots$. Now we construct a new real number as follows:

- Let this real number be between 0 and 1. So it is in the form of $0.????? \cdots$.
- Look at f(1)'s tenth(0.1) digit. If it is equal to 0, let the tenth digit of our number be 1. If it is not, let the hundredth digit be 0..
- Look at f(2)'s hundredth(0.01) digit. If it is 0, our number gets 1 for the hundredth digit. If it is not, our number gets 0.
- Look at f(3)'s thousandth(0.001) digit. If it is 0, our number gets 1 for the thousandth digit. If it is not, our number gets 0.
- Do the same thing for every n.

If we do this process, we get a new real number r with digits being either 0 or 1. Since f is onto, r must have a natural number n such that $f(n) = r$. However, this is impossible, because r and f(n) differs in the nth digit below 0. If f(n) has 0 at that digit, r has 1. If f(n) has any other number at that digit, r has 0. So this is a contradiction, and we cannot find an onto function from N to R. So $|N| < |R|$. This elegant argument is called the Cantor's Diagonalization.

So we have an infinite set larger than the set of natural numbers! In fact, we can find a set larger than $|R|$, a set larger than that set, and so on. There are many more interesting facts in set theory that are very surprising, but for this mini lecture I think this will be sufficient. If you have any questions or feedbacks, feel free to send me an e-mail.

## 6 Miscellaneous (Technical Issues)

- Size of sets are usually called cardinality.
- Usually set theorists include 0 in N, but we did not.
- One-to-one functions are also called injections, and onto functions are also called surjections. If a function is both injective and surjective, it is called a bijection.
- Actually we were quite wrongfully using the notation $\le$ and $\ge$ when we were talking about one-to-one functions and onto functions. It is much better to just say $|X| \le |Y|$ if there is an one-to-one function from X to Y and define $\ge$ similarly. This is because of the Cantor-Bernstein-Schroeder Theorem which states that if there is an injection from X to Y and an injection from Y to X then $|X| = |Y|$. So using this definition we have $|X| \le |Y|$ and $|X| \ge |Y|$ implies $|X| = |Y|$. This is not true for our definition unless we assume the Axiom of Choice, an extremely strong mathematical axiom.